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FUNCTIONS OF LEGENDRE'S TYPE.

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Introduction.

The class of functions defined by the expression

$$\frac{1}{2^n . n!} \frac{d^n}{dz^n} (z^2 - 1)^n,$$

which occurs frequently in investigations connected with questions in most of the branches of applied mathematics, is due to Legendre[1], who obtained it as a solution of the Differential equation

$$(1 - z^2) \frac{d^2y}{dz^2} - 2z \frac{dy}{dz} + n(n + 1) y = 0$$

and also identified it with the polynomial of the nth degree in z, which occurs as the coefficient of h^n in the expansion in ascending powers of h of

$$(1 - 2zh + h^2)^{-\frac{1}{2}}.$$

Likewise the class of functions defined by an expression of the form

$$\frac{d^n}{dz^n} [(z - a)^n (z - b)^n]$$

which satisfies a Differential equation of the second order, (which we shall call *the Equation of Legendre's type*) may also be studied with a view to obtain results which are in marked resemblance with those of Legendre. The solutions of the equation, which are Polynomials when n is a positive integer, or functions in general, when n is unrestricted, are called in this paper, *polynomials* or *functions* of *Legendre's type*.

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[1] *Memoirs par divers Savans* X (1875).

§ 1. *The Polynomials.*

Let us consider the expansion in powers of h of

$$[1-2(z-d)h/\gamma+h^2]^{-\frac{1}{2}}, \quad \ldots\ldots (1.1)$$

which could be expanded in powers of $2(z-d)h/\gamma+h^2$ if the modulus of that expression is less than unity. Owing to the absolute convergence of this, the expansion can be rearranged as power series in h, and we accordingly write

$$[1-2(z-d)h/\gamma+h^2]^{-\frac{1}{2}}=P_0(z+)P_1(z)h+\ldots+P_n(z)h^n+\ldots (1.2)$$

where $P_n(z)$ may be called by analogy, *polynomial* (of degree n) of Legendre's type.

Now if

$$(hd+\gamma)-hy=[h^2\gamma^2-2h\gamma(z-d)+\gamma^2]^{\frac{1}{2}} \ldots\ldots (1.3)$$

so that

$$y=z+h\frac{(y-d-\gamma)(y-d+\gamma)}{z\gamma} \ldots\ldots (1.31)$$

we have by Lagrange's theorem[2], the expansion

$$y=z+h\frac{(z-a)(z-b)}{a-b}+\ldots+\frac{h^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\frac{(z-a)^n(z-b)^n}{(a-b)^n}\right]+\ldots (1.32)$$

where we write $2d=a+b$ and $2\gamma=a-b$.

Differentiating this with respect to z, we obtain

$$\frac{dy}{dz}=\frac{\gamma}{[h^2\gamma^2-2h\gamma(z-d)+\gamma^2]^{\frac{1}{2}}}$$

$$=[1-2h(z-d)/\gamma+h^2]^{-\frac{1}{2}}$$

$$=1+h\frac{d}{dz}\left[\frac{(z-a)(z-b)}{a-b}\right]+\ldots+\frac{h^n}{n!}\frac{d^n}{dz^n}\left[\frac{(z-a)^n(z-b)^n}{(a-b)^n}\right]+\ldots (1.321)$$

Equating the coefficients of h^n in the two expansions (1.2) and (1.31) we find

$$\left.\begin{array}{l} P_n(z)=\dfrac{1}{n!}\dfrac{1}{c^n}\dfrac{d^n}{dz^n}[(z-a)^n(z-b)^n] \\ \text{where } c=a-b=2\gamma. \end{array}\right\} \ldots\ldots (1.4)$$

This corresponds to Rodrigue's formula[3] for the usual Legendre's polynomials, (quoted in the introduction); and supplies the easiest way of calculating the actual values of $P_n(z)$ which, it will be noticed, are complete homogeneous polynomials of degree n in z, a and b.

Now the equation $(z-a)^n(z-b)^n=0$ has $2n$ real roots (a and b being supposed real) of which n may be supposed to coincide at $z=a$,

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[2] Edwards *Differential Calculus*, § 517.

[3] *Cor. resp. sur l' école polytechnique*, III (1814—16) pp. 361—85.

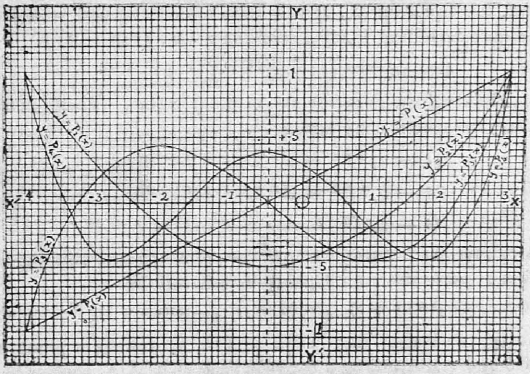
and the n others at $z = b$. By a well-known theorem, the first derived equation

$$\frac{d}{dz}\left[(z-a)^n (z-b)^n\right] = 0$$

will have $2n - 1$ real roots separating those of the original equation. Passing to the nth derived equation we find that

$$\frac{d^n}{dz^n}\left[(z-a)^n (z-b)^n\right] = 0$$

has n real roots and that these must all lie between a and b. The roots are all separate ; for, two roots could only be coincident if the original equation $(z-a)^n (z-b)^n = 0$ had $n+1$ coincident roots. Thus the n roots of $P_n(z) = 0$ are all real and separate and lie between $z = a$ and $z = b$. The accompanying figure shows the graphs of $P_1(z)$; $P_2(z)$; $P_3(z)$ and $P_4(z)$ drawn for the value $a = 3$, $b = -4$ between the values $(-4, 3)$ for z.



§ 2. *The Differential Equation.*

It is very convenient to express $P_n(z)$ in the form of an integral corresponding to Schläfli's[*] integral for Legendre's polynomials.

With the aid of Cauchy's theorem and (1 . 4) it follows at once that

$$P_n(z) = \frac{1}{2\pi i}\int_C \frac{(t-a)^n (t-b)^n}{c^n (z-t)^{n+1}}\, dt \qquad (2 . 1)$$

where C is a contour in the t-plane encircling the point z once counter-

* Schlafli : *Heine'schen Kugelfunktionen* (1881).

clockwise. This last integral furnishes an easy proof that $P_n(z)$ is a solution of the Differential equation

$$(z-a)(z-b)\frac{d^2u}{dz^2}+(2z-a-b)\frac{du}{dz}-n(n+1)\,u=0 \qquad (2.2)$$

Substituting from (2 . 1) in (2 . 2) we have

$$(z-a)(z-b)\frac{d^2P_n}{dz^2}+(2z-a-b)\frac{dP_n}{dz}-n(n+1)\,P_n$$

$$=\frac{(n+1)}{c^n.2\pi i}\int_C\frac{(t-a)^n(t-b)^n}{(t-z)^{n+3}}\left[(n+2)(z-a)(z-b)\right.$$

$$\left.+(2z-a-b)(t-z)-n(t-z)^2\right]dt$$

$$=-\frac{(n+1)}{c^n.2\pi i}\int_C\frac{d}{dt}\left[\frac{(t-a)^{n+1}(t-a)^{n+1}}{(t-z)^{n+2}}\right]dt \qquad (2.21)$$

which vanishes, since $(t-a)^{n+1}(t-b)^{n+1}(t-z)^{-n-2}$ resumes its original value after describing C when n is an integer. Hence $P_n(z)$ is therefore a solution of the differential equation (2 . 2). Now when n is unrestricted, the equation (2 . 2) will have a meaning and will be satisfied also by the corresponding $P_n(z)$ as defined in (2 . 1) if

$$(t-a)^{n+1}(t-b)^{n+1}(t-z)^{-n-2}$$

though multiple-valued resumes its original value. These many-valued functions may be specified by taking

$$(t-a)^{n+1}=e^{(n+1)\log(t-a)}$$

$$(t-b)^{n+1}=e^{(n+1)\log(t-b)}$$

$$(t-z)^{-n-2}=e^{-(n+2)\log(t-z)}$$

so that as t describes a contour round the point a, b or z counter-clockwise, the function repeats itself multiplied respectively by

$$e^{2\pi i(n+1)},\ e^{2\pi i(n+1)},\ e^{-2\pi i(n+2)}.$$

Consequently, whatever may be a and b, by taking a point A beyond a and z or b and z and taking a contour such that at A, the

$$|\arg(t-z)|\,;\ |\arg(t-a)|\,;\ |\arg(t-b)|$$

are less than π, and such that it starts from A, encircles the point a in the positive direction, and returns to A, and then encircles the point z in the positive direction and returns to P we see that

$$(t-a)^{n+1}(t-b)^{n+1}(t-z)^{-n-2}$$

$$=\exp.\left[(n+1)\log(t-a)+(n+1)\log(t-b)-(n+2)\log(t-z)\right]$$
$$\times\exp.\left[(n+1)\,2\pi i-(n+2).\,2\pi i\right]$$
$$=(t-a)^{n+1}(t-b)^{n+1}(t-z)^{-n-2}$$

that is to say, its original value.

Hence the expression (2 . 21) is equal to zero, and therefore

$$P_n(z) = \frac{1}{2\pi i \,.\, c^n} \int_A^{(a+,\, z+)} \frac{(t-a)^n (t-b)^n}{(t-z)^{n+1}}\, dt \qquad (2\ 3)$$

satisfies for all values of n, the equation (2 . 2) . $P_n(z)$ so defined, will not then be a Polynomial, but be an infinite series which we may agree to call a *function of Legendre's type.*

Again reverting to the equation (2 . 2) which may be written

$$\frac{d^2u}{dz^2} + \left(\frac{1}{z-a} + \frac{1}{z-b}\right)\frac{du}{dz} - \frac{n(n+1)}{(z-a)(z-b)}\, u = 0.$$

We easily notice that it is a particular case of Riemann's P-function[5] defined by the scheme

$$P_n(z) = P\begin{Bmatrix} a & b & \infty & \\ 0 & 0 & n+1 & z \\ 0 & 0 & -n & \end{Bmatrix}. \qquad (2\,.\,4)$$

Since the equation (2 . 2) is a particular case of Riemann's equation, it is natural to expect that $P_n(z)$ can be expressed in terms of hypergeometric functions. In (2 . 1) or (2 . 3) we choose the contour C such that $\left|\frac{t-z}{t-a}\right| < 1$, so that $(t-z)^{-n-1}$ can be expanded in powers of $(z-a)$ in an uniformly convergent series, *viz.*,

$$\frac{1}{(t-z)^{n+1}} = \frac{1}{(t-a)^{n+1}}\left\{1 + \sum_{r=0}^{\infty} \frac{(n+1)\,(n+2)\ldots(n+r)}{1\,.\,2\,.\,3\ldots r}\left(\frac{z-a}{t-a}\right)^r\right\}.$$

Let us take for sake of simplicity, that a and b are real so that $\arg\,(t-b)=0$ at $t=a$. Integrating term by term the expression obtained by substituting (2 . 5) in (2 . 1) we have

$$P_n(z) = \frac{1}{c^n\,.\,2\pi i}\sum_{r=0}^{\infty}\frac{(z-a)^r}{r!}(n+1)\ldots(n+r)\int_C \frac{(t-a)^n\,(t-b)^n}{(t-a)^{n+1+r}}\,dt$$

$$= \frac{1}{c^n\,.\,2\pi i}\sum_{r=0}^{\infty}\frac{(z-a)^r}{r!}(n+1)\ldots(n+r)\int_C \frac{(t-b)^n}{(t-a)^{r+1}}\,dt$$

$$= \frac{(z-a)^r}{c^n}\,.\,\frac{(n+1)(n+2)\ldots(n+r)}{(r!)^2}\left[\frac{d^r}{dt^r}(t-b)^n\right]_{t=a}$$

$$= (-)^r\left(\frac{z-a}{a-b}\right)^r \frac{(n+1)\ldots(n+r)}{1\,.\,2\ldots r}\,.\,\frac{(-n)(-n-1)\ldots(-n-r-1}{1\,.\,2\ldots r}$$

$$= F\left[(n+1),\, -n,\, 1,\, \frac{z-a}{b-a}\right] \qquad \ldots \qquad \ldots \quad (2\,.\,6)$$

where F is the hypergeometric function.

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[5] Papperitz: *Math, Annalen* (1885), Vol. 35, p. 213.
Riemann: *Gessell, d. Wiss. Zu Göttingen*, VII (1857).

In this form we see at once that

$$P_n(z) = P_{-(n+1)}(z) \quad \ldots \quad \ldots \quad \ldots \quad (2.61)$$

Also from the hypergeometric equation (2 . 2) it is obvious[6] that it has a solution of the type

$$(z-a)^{\alpha}(z-b)^{\beta}(z-c)^{\gamma}\int_C (t-a)^{\beta+\gamma'+\alpha'-1}(t-b)^{\gamma+\alpha+\beta'-1}$$

$$(t-c)^{\alpha+\beta+\gamma'-1}(z-t)^{-\alpha-\beta-\gamma}\,dt$$

along a proper contour C in the t-plane, where $\alpha=0=\beta'=\alpha'=\beta'$, $\gamma=n+1$, $\gamma'=-n$ and $c=\infty$.

Then we shall have as a solution of (2 . 2)

$$\int_C (t-a)^n(t-b)^n(t-z)^{-n-1}\,dt$$

which is the integral (2 . 1), so that the Differential equation or the hypergeometric series, may also be used as the starting point of the theory of this subject.

Or yet again, following Dr. Barnes[7] we may start with the integral

$$\frac{1}{2\pi i}\int_{-\infty i}^{\infty i} ds\,\frac{\Gamma(s-n)\Gamma(s+n+1)\Gamma(-s)}{\Gamma(1+s)}\left(\frac{a-z}{b-a}\right)^s$$

where a, b being real, $|\arg(-z)|<\pi$, and the path of integration is such as to ensure (curved if necessary) that the poles of $\Gamma(s-n)$ and $\Gamma(s+n+1)$ viz.,

$$s = n,\ n-1,\ n-2,\ldots$$

$$\text{and}\quad s = -n-1,\ -n-2,\ -n-3,\ldots$$

lie on the left of the path, and the poles of $\Gamma(-s)$, viz, $s=0, 1, 2,\ldots$ lie on the right of path. In the case however when n is a positive integer, this is not possible for the hypergeometric series reduces to a Polynomial.

§ 3. *Recurrence Formulæ.*

We proceed to establish a number of formulæ connecting these functions of different degrees. These can be obtained by any of the methods employed in defining them, either (1 . 2), (2 . 1) or (2 . 7). Employing however (2 . 1), we notice that

$$\frac{1}{2\pi i}\int_C \frac{d}{dt}\left[\frac{(t-a)^{n+1}(t-b)^{n+1}}{(t-z)^{n+1}}\right]dt \quad \ldots \quad (2.1)$$

[6] Pochhammer: *Math. Annalen*, Vol. XXXV. Hobson: *Phil. Trans. Roy. Soc.* A. (1871).

[7] E. W. Barnes: *Proc. Lond. Math. Soc.* (2) Vol. 6. pp. 141—77. Also *Quart Jour. of Math.*, Vol. 39, pp. 97—204.

taken around a simple contour in the t-plane enclosing the point z, vanishes if the function enclosed in [ ] is single-valued—which as we noticed above is the case for all values of n, if the contour describes only one of the two points a and b once in the counter-clockwise direction. Under these assumptions then, we obtain by expanding (3 . 1)

$$\int_C \frac{(t-a)^{n+1}(t-b)^{n+1}}{(t-z)^{n+2}}\,dt = \int_C \frac{2(t-d)}{(t-z)^{n+1}}(t-a)^n(t-b)^n\,dt$$

$$\text{i.e.,}\quad P_{n+1}(z) = \frac{1}{c^{n+1}.\,2\pi i}\int_C \frac{2(t-z)(t-a)^n(t-b)^n}{(t-z)^{n+1}}\,dt$$

$$+\frac{1}{c^{n+1}.\,2\pi i}\int_C \frac{2(z-d)(t-a)^n(t-b)^n}{(t-z)^{n+1}}\,dt \quad \text{from (2 . 1)}$$

$$\text{i.e.,}\quad cP_{n+1}(z) - 2(z-d)\,P_n(z) = \frac{1}{c^n.\,2\pi i}\int_C \frac{(t-a)^n(t-b)^n}{(t-z)^n}\,dt.$$

Differentiating with respect to z, and transposing

$$cP'_{n+1}(z) - 2(z-d)\,P'_n(z) = 2(n+1)\,P_n(z) \qquad \ldots \quad (3.11)$$

which connects P_{n+1} with P_n and its derivate.

Similarly, starting with

$$\frac{1}{2\pi i}\int_C \frac{d}{dt}\left[\frac{t(t-a)^n(t-b)^n}{(t-z)^n}\right]dt \qquad \ldots \quad (3.2)$$

instead of (3 . 1) we obtain by substituting from (2 . 1) and (3 . 11),

$$c(n+1)\,P_{n+1}(z) - 2(2n+1)(z-d)\,P_n(z) + nc\,P_{n-1}(z) = 0, \quad (3.21)$$

which connects three successive functions.

Differentiating (3 . 21) and employing (3 . 11), we obtain

$$2n\,P_n(z) + cP'_{n-1}(z) - 2(z-d)\,P'_n(z) = 0 \qquad \ldots \quad (3.3)$$

Similarly, we have

$$cP'_{n+1}(z) - cP'_{n-1}(z) = 2(2n+1)\,P_n(z) \qquad \ldots \quad (3.4)$$

The formulæ (3 . 11), (3 . 21), (3 . 3) and (3 . 4) are the recurrence formulæ which we desired to establish.

§ 4. *Integral properties. Expansion of any arbitrary function.*

The formulæ (1 . 4) will now be employed to get some properties of integrals which involve in the integrand these functions. Thus

$$\int_a^b P_m(z)\,P_n(z)\,dz = \int_a^b \frac{1}{\Gamma(m+1)}\cdot\frac{1}{\Gamma(n+1)}\cdot\frac{1}{c^{m+n}}\,\underset{n}{U}^{n}\,\underset{m}{U}^{m}\,dz$$

where $\underset{n}{U}^{r} = \frac{d^n}{dz^n}\left[(z-a)^r(z-b)^r\right]$.

Now assuming m and n to be integers, and integrating by parts, the expression equals

$$\mathrm{K}\,\Big[\mathrm{U}^{n}_{n-1}\,\mathrm{U}^{m}_{m}\Big]_a^b - \mathrm{K}\int_a^b \mathrm{U}^{m}_{n-1}\,\mathrm{U}^{m}_{m+1}\,dz,$$

$$\Big[\mathrm{K} = 1/c^{m+n}\,\Gamma(m+1)\,\Gamma(n+1),$$

$$= -\mathrm{K}\int_a^b \mathrm{U}^{n}_{n-1}\,\mathrm{U}^{m}_{m+1}\,dz,$$

since the other vanishes at both limits

$$= (-)^n\,\mathrm{K}\int_a^b (z-a)^n\,(z-b)^n\,\mathrm{U}^{m}_{m+n}\,dz, \text{ ultimately.}$$

Now, when $n > m$, $\dfrac{d^{m+n}}{dz^{m+n}}\left[(z-a)^m\,(z-b)^m\right]$

obviously vanishes. The same reasoning applies also when $m > n$. Accordingly the integral vanishes when m is not equal to n.

On the other hand if $m = n$, the last step is untenable, but will become instead $= (-)^n\,\mathrm{K}\int_a^b (z-a)^n\,(z-b)^n\,(2n)!\,dz$

which on putting $z = a\cos^2\theta + b\sin^2\theta$

$$= \frac{2n!}{n!\,n!\,c^{2n}}\int_0^{\pi/2} 2\,.\,c^{2n+1}\,\sin^{2n+1}\theta\,\cos^{2n+1}\theta\,d\theta = \frac{a-b}{2n+1}.$$

$$\left.\begin{aligned}\text{Consequently } &\int_a^b \mathrm{P}_m(z)\,\mathrm{P}_n(z)\,dz\\ &= 0, \text{ if } m \text{ is not equal to } n\\ &= (a-b)/(2n+1), \text{ if } m = n\end{aligned}\right\} \quad \ldots \quad (4\,.\,1)$$

which establishes the orthogonal[8] property of the function. This property is true in fact for all values of n, real or complex, except $n = -p - \frac{1}{2}$, where p is zero or a positive integer; and can be established easily by employing with the definition (2 . 6) and the method of Barnes.[9]

Now the result of (4 . 1) is used among other things for obtaining the expansion of any Polynomial in a series of these Polynomials of Legendre's type. For if $\mathrm{F}(z)$ is any Polynomial of degree n in z, it is always possible to choose $a_0, a_1 \ldots a_n$ such that $\mathrm{F}(z) = a_0\mathrm{P}_0(z) + a_1\mathrm{P}_1(z) + \ldots + a_n\mathrm{P}_n(z)$. To determine a_r easily, we multiply both sides by $\mathrm{P}_r(z)$ and integrate between the limits a and b. Thus

$$\int_a^b \mathrm{P}_r(z)\,\mathrm{F}(z)\,dr = \sum_{s=1}^{n}{}' \int_a^b a_0\mathrm{P}_r(z)\,\mathrm{P}_s(z)\,dz + \int_a^b a_r[\mathrm{P}_r(z)]^2\,dz$$

$$= a_r\,\frac{a-b}{2n+1}$$

where the dash (′) on Σ indicates the omission of $s = r$.

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[8] Murphy: *Camb. Phil. Trans.*, Vols., IV and V.

[9] *Quart. Jour. of Math.*, Vol. 39, p. 186, § 65.

Consequently, $a_r = \dfrac{(2n+1)}{a-b} \int_a^b P_r'(z)\, F(z)\, dz$ ... (4.2)

which implies the integration of $\int_a^b z^k P_r(z)\, dz$ and this can be done in terms of gamma functions.

§ 5. *Functions of Hermite's type and their connection with functions of Legendre's type.*

The connection between the functions of Legendre and those of Hermite[10] defined for positive integral values of n by the relation

$$H_n(z) = e^{z^2} \frac{d^n}{dz^n} \left(e^{-z^2}\right)$$

has been worked out at some length by H. E. J. Curzon[11]. It is interesting as a continuation of the preceding analysis to establish similar connection between the functions of Legendre's type and those defined for positive integral values of n by the relation

$$H\ (z) = e^{(z-a)(z-b)} \frac{d^n}{dz^n} \left[e^{-(z-a)(z-b)} \right] \qquad (5.1)$$

which we may call functions of Hermite's type. When n is unrestricted, it is easy to see that the same function may be defined by the relation

$$H_n(z) = \frac{\Gamma(n+1)}{2\pi i} \int_C \frac{e^{(z-a)(z-b)-(t-a)(t-b)}}{(t-z)^{n+1}}\, dt \qquad (5.11)$$

where C is a suitable contour. Transforming the pole to the origin, the same may also be written

$$H_n(z) = \frac{\Gamma(n+1)}{2\pi i} \int_C t^{-n-1}\, e^{-2(z-d)t - t^2}\, dt. \qquad (5.12)$$

From this formula we see at once that

$$\frac{d}{dz} H_n\ (z) = -2n\, H_{n-1}(z) \qquad (5.13)$$

and that $H_n(z)$ satisfies the differential equation

$$\frac{d^2u}{dz^2} - 2(z-d)\frac{du}{dz} + 2nu = 0. \qquad (5.131)$$

Since $\int^\infty e^{-t^2 + \zeta t}\, t^{-N-1}\, dt$

is convergent for all finite values of ζ and N, it is easy to show that it is permissible to expand $e^{-2(z-d)t}$ in powers of $(z-d)$. Thus

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[10] Hermite. *Compte Rendus t.* 58, pp. 266—273.

[11] H. E. J. Curzon: *Proc. Lond. Math. Soc.* (2) Vol, 12, pp. 236—59.

$H_n(z)$

$$= \frac{\Gamma(n+1)}{2\pi i} \int_C \sum_0^\infty \frac{(-\overline{2z-d})^r}{r!} e^{-t^2} t^{-n-1+r} dt$$

$$= \frac{\Gamma(n+1)}{2\pi i} \sum_0^\infty \frac{(-2y)^r}{r!} i \; e^{-(n-r)\pi i} . \sin(-\overline{n-r}\pi) \Gamma\left(\frac{r-n}{2}\right)$$

$$= \frac{\Gamma(n+1)}{2\pi} . e^{-n\pi i} \sum_0^\infty \frac{(2y)^r}{r!} 2\cos\left(-\frac{n-r}{2}\pi\right) \sin\left(-\frac{n-r}{2}\pi\right) \Gamma\left(\frac{r-n}{2}\right)$$

$$= \frac{\Gamma(n+1)}{2\pi} . e^{-n\pi i} \sum_0^\infty \frac{(2y)^r}{r!} \cos\left(\frac{n-r}{2}\pi\right) \Big/ \Gamma\left(\frac{n-r}{2}+1\right)$$

$$= e^{-n\pi i} \Gamma(n+1) \sum_0^\infty \left(\frac{(2y)^{2s}}{2s!}\right) \cos\frac{n\pi}{2} (-)^s \Big/ \Gamma\left(\frac{n}{2}-s+1\right)$$

$$+ e^{-n\pi i} \Gamma(n+1) \sum_0^\infty \frac{(2y)^{2s+1}}{2s+1} \sin\frac{n\pi}{2} (-)^s \Big/ \Gamma\left(\frac{n}{2}-s+\tfrac{1}{2}\right) \quad (5.2)$$

where $y = (z - d)$.

Now if n is a positive integer, one of these vanishes and the second vanish after $s = \frac{n}{2}$.

$$\text{Thus } H_n(z) = \sum_0^{s=n/2} \frac{2^{2s} (z-d)^{2s}}{(2s)!} . \frac{n!}{(n/2-s)!} (-)^{n/2-s} \quad (5.21)$$

when n is an even positive integer.

We proceed to express $H_n(z)$ in terms of $P_n(z)$ and *vice versa.*

Now when n is an even positive integer.

$$\int_{-\infty}^{\infty} e^{-t^2} H_n(ty/\gamma)dt, = \int_{-\infty}^{\infty} e^{-t^2} t^{2s} \sum_{s=0}^{n/2} \frac{n!}{2s!} \left(\frac{2y}{\gamma}\right)^{2s} \frac{(-)^{n/2-s}}{(n/2-s)!} dt$$

$$= \sum_{s=0}^{n/2} \frac{\sqrt{\pi}(2s)!}{2^{2s} . s!} . \frac{n!}{2s!} \left(\frac{2y}{\gamma}\right)^{2s} \frac{(-)^{n/2-s}}{(n/2-s)!}$$

$$= \sum_{s=0}^{n/2} \frac{\sqrt{\pi}\, n!}{(n/2)!} \quad \frac{(n/2)!\,(-)^{n/2-s}}{s!\,(n/2-s)!} \left(\frac{y}{\gamma}\right)^{2s}$$

$$= \frac{\sqrt{\pi} . n!}{(n/2)!\, \gamma^n} (y^2 - \gamma^2)^{n/2}$$

$$= \frac{n!\, \sqrt{\pi}}{\gamma^n . (n/2)!} (z-a)^{n/2} (z-b)^{n/2}. \quad (5.3)$$

Employing (5 . 12) and differentiating (5 . 3) $n/2$ times we have,

$$\frac{1}{\gamma^n}\frac{(-z)^{n/2}\, n!}{(n/2)!}\int_{-\infty}^{\infty} e^{-t^2}\, t^{n/2}\, H_n\left[\frac{yt}{\gamma}\right] dt$$

$$=\frac{n!}{\gamma^n}\frac{\sqrt{\pi}}{(n/2)!}\frac{d^{n/2}}{dz^{n/2}}\left[(z-a)^{n/2}(z-b)^{n/2}\right]$$

Replacing $\frac{n}{2}$ by m, and substituting from (1 . 4) for $P_m(z)$

$$(-)^m\int_{-\infty}^{\infty} e^{-t^2}\, t^m\, H_m\left[\frac{yt}{\gamma}\right] dt = \sqrt{\pi}\,\gamma^m . m!\; P_m(z)$$

or,
$$P_m(z)=\frac{(-)^m}{\sqrt{\pi} . m!}\int_{-\infty}^{\infty} e^{-t^2}\left[\frac{t}{\gamma}\right]^m H_m\left[\frac{yt}{\gamma}\right] dt$$

$$=\frac{\gamma}{\sqrt{\pi} . m!}\int_{-\infty}^{\infty} e^{-\gamma^2 t^2}\, t^m\, H_m(yt)\, dt. \tag{5·4}$$

This is the relation that we desired to establish which is now shown to hold for positive integral values of n. The remainder shall be taken up to establish the same with some modifications when n is unrestricted.

We shall have to show first that

$$U = A\int_C e^{-\gamma^2 t^2}\, t^n\, H_n\,[t\,(z-d)]\, dt \tag{5.41}$$

is a solution of the differential equation (2 . 2); and secondly, that $P_n(z)$ and $P'_n(z)$ for some assigned value of z, are equivalent to the values of $U(z)$ and $U'(z)$ for the same values of z. To show that the first is true, we differentiate and substitute for U in the Differential equation. Thus we have to show that

$$\int e^{-\gamma^2 t^2}\, t^n\left[t^2(z-a)\,(z-b)\,H''_n+2(z-d)\,tH'_n-n(n+1)\,H\right] \tag{5.411}$$

taken along a bulb-like contour extending to infinity is equal to zero. Now

$$\frac{d}{dt}\left[e^{-t^2}\, t^{n+1}\left\{ nH_n(\overline{z-d}\,.\,t)-(z-d)\,t\,H'_n(z-d)\,t^2\right\}\right]$$

$$=\left[-t^2H''_n(z-a)\,(z-b)-2(z-d)\,tH'_n+n(n+1)H_n\right]\times{}^{-t^2}\,t^n$$
$$+\;\gamma^2 t^2\left[H''_n-2(z-d)\,H'_n+2n\,H_n\right]e^{-t^2}\,t^n.$$

But since H_n satisfies the equation (5 . 131) the expression in the second line vanishes, and that in the first line is the same as the expression in [ ] in (5 . 411). Consequently we have to show that

$$\int\frac{d}{dt}\left[e^{-\gamma^2 t^2}\, t^{n+1}\left\{ nH_n\,(z-d)\,t-(z-d)\,t\,H'_n\,(z-d)\,t\right\}\right] dt$$

taken along the bulb-contour extending to infinity vanishes; in other words, we have to show that

$$\lim_{t \to a} e^{-\gamma^2 t^2} t^{n+1} [n H_n \overline{z-d}\, t - (z-d)\, t\, H'_n (z-d)\, t]$$

tends to zero.

Now it is not very difficult to show that

$$| H_n \{ (z-d)t \} |$$

and $$| H'_n \{ (z-d)t \} | = 2n \,| H_{n-1} \{ (z-d)t \} |$$

are of the order $\zeta^{2\gamma} e^{\zeta^2}$, where $\zeta = | z - d |$.

Consequently, the limit we are desiring to evaluate will be zero, if $\zeta^2 - \gamma^2$ is negative; *i.e.*, if $| z - d | < \gamma$.

Hence, when $| z - d | < \gamma$, $\int_C e^{-\gamma^2 t^2} t^n H_n [(z-d)\, t]\, dt$ is a solution of the equation of Legendre's type, *viz.* (2.2)

Now in order to completely identify P_n in terms of H_n, we have to properly choose the constant A appearing in (5.41); in fact,

$$y = \frac{\gamma^{n+1}}{\sqrt{-\pi}\, \sin n\pi\, \Gamma(n+1)} \int e^{-\gamma^2 t^2} t^n H_n [(z-d)\, t]\, dt \qquad (5.5)$$

tends to $P_n(d)$ and H_n tends to the value $H_n(0)$.

Likewise $P'_n(d) = [y']_{z=d}$.

To show this, we notice that from (5.2)

$$H_n(0) = \frac{\Gamma(n+1) \cos \frac{n\pi}{2}}{e^{n\pi i}\, \Gamma\left(\frac{n}{2} + 1\right)} \quad \ldots \quad \ldots \qquad (5.51)$$

Also $\int_C e^{-\gamma^2 t^2} t^n\, dt = \dfrac{i\, e^{n\pi i} \sin n\pi}{\gamma^{n+1}}\, \Gamma\left(\dfrac{n+1}{2}\right)$.[12]

Therefore the right side of (5.5)

$$= \frac{\cos \frac{n\pi}{2}\, \Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{n}{2} + 1\right)}$$

$$= F\left(-\frac{n}{2}, \frac{n+1}{2}, 1, 1\right)$$[13]

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[12] Whittaker and Watson; *Modern Analysis*, § 12, 22.

[13] *Ibid.*, § 14, 11.

Also from (2 . 6)

$$P_n(d) = F(-n,\ n+1,\ 1,\ \tfrac{1}{2})$$
$$= F\left(-\frac{n}{2},\ \frac{n+1}{2},\ 1,\ 1\right) \text{ by Kummer's formula.}^{14}$$

Consequently $P_n(d) = [y]_{z=d}$.

Again differentiating (5 . 5) and employing (5 . 12) that

$$H'_n = -2n\, H_{n-1},$$

we obtain

$$H'_n(0) = 2\,\frac{\Gamma(n+1)\sin\frac{n\pi}{2}}{e^{n\pi i}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}.$$

Hence the right side of (5 . 5) after differentiation is equal to

$$\frac{\gamma^{n+1}}{\sqrt{-\pi}\cdot\Gamma(n+1)\sin n\pi} \times \frac{2\Gamma(n+1)\sin\frac{n\pi}{2}}{e^{n\pi i}\,\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)} \times \frac{i\,e^{n\pi i}\sin n\pi}{\gamma^{n+2}} \times \Gamma\left(\frac{n}{2}+1\right)$$

$$= 2\,\frac{\Gamma\left(\frac{n}{2}+1\right)\sin\frac{n\pi}{2}}{\gamma\sqrt{\pi}\,.\,\Gamma\left(\frac{n+1}{2}\right)}.$$

To identify it with the value of $P'_n(d)$ we use (3 . 11)—in which we write $(n-1)$ for n — and obtain $\gamma P'_n(d) = n\,P_{n-1}(d)$,

or,
$$P'_n(d) = 2\,\frac{\Gamma\left(\frac{n}{2}+1\right)\sin\frac{n\pi}{2}}{\gamma\sqrt{\pi}\ \Gamma\left(\frac{n+1}{2}\right)},$$

as in the preceding; thus completely identifying (5 . 5) as a solution of (2 . 2).

In other words, throughout the domain $|z-d| < \gamma$, we have

$$P_n(z) = \frac{\gamma^{n+1}}{i\sqrt{\pi}\,\Gamma(n+1)\,.\sin n\pi}\int_0 e^{-\gamma^2 t^2}\, t^n\, H_n[(z-d)\,t]\,dt. \qquad (5\,.\,6).$$

In conclusion, we may merely observe that this analysis may be extended to express H_n in terms of P_n, as also to obtain the second solutions of the equations (2 . 2) of Legendre and (5 . 14) of Hermite and establish the reciprocal relationships among the four functions.

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[14] *Ibid.*, p. 292, Ex. 12.

MORLEY'S THEOREM.

By M. Bhimasena Rao.

Introduction.

This theorem was first brought to the notice of the Indian reader by Mr. J. H. Grace, M.A., F.R.S., in his paper, "The Nine-point Circle" published in the Journal of the Indian Mathematical Society, Vol. IX, page 97, with the remark "A singular extension of Feuerbach's Theorem of which I have seen only the enunciation." The present paper contains two proofs of the Theorem. The first is a general proof depending on the following result :—

If a pencil of cubics passes through the six points of intersection of four lines and through three more points, P, Q, R, the tangents at P, Q, R to each member of the pencil form a triangle in perspective with PQR.*

The second proof is based on the following property which, though deducible from the above result, is proved independently by making use of known theorems regarding circular cubics :—

If a pencil of circular cubics passes through six points lying three by three on four lines, the normal at the ninth common point to each member of the pencil passes through its double focus.

In both the proofs is employed the well-known theorem in Conic Sections, *viz.*:

The envelope of the polars of a given point with respect to a range of conics is a conic.

Finally the argument is made to rest upon the theorem that the envelope of all cubics which pass through six points and touch two lines, say at a variable point P on the one, and at its associated point Q on the other, coincides with the locus of the ninth common point of intersection of any two cubics through the above eight points.

Part I.

Coming to the theorem itself, the enunciation of the same as given by Mr. Grace, is as follows :—

Every curve of class three which touches the six joins of four orthocentric points and passes through the circular points touches the nine-point circle.

We shall establish the truth of the reciprocal of this theorem, *viz.*:

Every cubic curve which passes through the six points of intersection of

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* The three points P, Q, R are not all arbitrary. They are so chosen as to admit of a pencil of cubics to pass through. However, if any two of them are given, the third is determined uniquely. For a geometrical construction of the same, see Cayley: *Collected Mathematical Papers*, Vol. IV, pp. 495—504.

four lines and touches the double lines of the pencil in involution determined at any point by the range of conics touching the four lines, touches a conic.

To this end, let us first consider the pencil of cubics that could be drawn through eight points A, B, C, A′, B′, C′, P, Q. It is known that these cubics pass through a ninth commom point, say R. Moreover two cubics intersect in nine points which is also the number required to completely specify a cubic. Hence a variable cubic passing through six points and touching two lines, say at a variable point P on the one, and its associated point Q on the other, intersects with its consecutive position, at the ninth common point of intersection of the pencil of cubics through the above eight points R is therefore the point of contact of the cubic with its envelope.

From another point of view, R is also the common point of intersection of the four conics ABCPQ, AB′C′PQ, BC′A′PQ and CA′B′PQ, provided A′,B′,C′ are collinear and lie respectively on BC, CA, AB.*

When P and Q are circular points at infinity we have the well-known theorem that the circum-circles of the four triangles formed by the four lines have a common point.

Now take one of these conics, say ABCPQR. Two triangles ABC and PQR are inscribed in it. It is well-known that there exists a conic, say F, which touches the six lines, AB, BC, CA. PQ, QR, RP. Also, as AB′C′ and PQR are inscribed in the conic AB′C′PQR, a conic can be drawn to touch PQ, QR, RP, B′C′, AB′ *i.e.* AC and AC′ *i.e.* AB ; that is to say, five of the lines which F touches. Hence this conic must be identical with F. In other words, there exists a conic which touches the seven lines PQ, QR, RP, AB, BC, CA and A′B′C′.

Now let us consider F in relation to the conic ABCPQR. There must be a conic which reciprocates F into ABCPQR. Call this conic S. With respect to S, then, ABC and PQR are self-conjugate triangles. PQ is the polar of R with respect to S. Referred to the triangle ABC, let x,y,z be the co-ordinates of R, and x',y',z' be the co-ordinates of any point R′ on PQ. Then S is the locus of the points $(\sqrt{xx'}, \pm\sqrt{yy'}, \pm\sqrt{zz'})$.†

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* These conics together with the lines BC, CA, AB, A′ B′ C′, are degenerate forms of cubics through the eight points, A, B, C, A′, B′, C′, P, Q, and hence pass through the ninth common points.

† The conic S being self-conjugate *w. r. t.* ABC, may be taken as

$$p\alpha^2 + q\beta^2 + r\gamma^2 = 0. \qquad \ldots \qquad \ldots \quad \text{(i)}$$

The polar of R (x, y, z) *w. r. t.* S is

$$px\alpha + qy\beta + rz\gamma = 0. \qquad \ldots \qquad \ldots \quad \text{(ii)}$$

Since R′ (x', y', z') lies on (ii), we have

$$pxx' + qyy' + rzz' = 0;$$

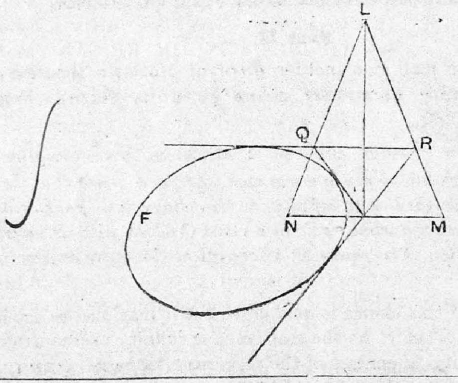
showing that the points $(\sqrt{xx'}, \pm\sqrt{yy'}, \pm\sqrt{zz'})$ lie on S.

Now if R′ be given as the third point of intersection of PQ with a particular member of the pencil of cubics, the four pairs of points, *viz.*, AA′, BB′, CC′ and RR′ are conjugate points with respect to the pencil of conics through $(\sqrt{xx'}, \pm\sqrt{yy'}, \pm\sqrt{zz'})$ and the cubic passing, as it does, through these pairs of conjugate points, is its own conjugate with respect to this pencil of conics. The cubic therefore passes through P′ and Q′, the conjugates of P and Q with respect to this pencil. P′ and Q′ are respectively on QR and RP in a line* with R′, this line being the transform of the conic ABCPQR; that is to say, the locus of conjugates of points on the conic ABCPQR with respect to the above pencil of conics. Since the cubic cuts the sides of the triangle PQR in P′,Q′,R′ which are collinear, the tangent at P, Q, R form a triangle, say LMN, in perspective with P,Q,R.

We have now established the result, *viz.*:

If a pencil of cubics passes through the six points of intersection of four lines and through three more points P, Q, R the tangents at P,Q,R to each member of the pencil form a triangle in perspective with PQR.†

With the aid of the above result, the fundamental theorem can be demonstrated as follows :—



* This touches also the conic F.

† The equation of a cubic passing through the six points of intersection of four lines is, when referred to the triangle formed by three of the lines, of the form,

$$\frac{f}{\alpha}+\frac{g}{\beta}+\frac{h}{\gamma}+\frac{k}{p\alpha+q\beta+r\gamma}=0.$$

where $p\alpha+q\beta+r\gamma=0$ is the equation of the fourth line. From this it can be seen that the conic $\frac{f}{\alpha}+\frac{g}{\beta}+\frac{h}{\gamma}=0$ touches the cubic at the vertices of the triangle of reference, the point (f, g, h) being the centre of perspective of the triangle of reference and that formed by the tangents at its vertices.

Since LMN is in perspective with PQR, the lines PL, PN are harmonic conjugates with respect to PR, PQ which are tangents to F. PL, PN are therefore conjugate lines with respect to F. If NL, NM are the double lines of the pencil in involution determined at N by the range of the inconics of the quadrilateral formed by the lines AB, BC, CA and A′B′C′, F being one of such inconics, NL, LM are conjugate lines with respect to F. PL, PN, as also NL, NM being pairs of conjugate lines, it follows that MN is the polar of L with respect to F, and similarly NL is the polar of M. The triangle LMN is therefore self-conjugate with respect to F. The diagonal triangle of the quadrilateral referred to above is also self-conjugate with respect to F, and hence an inconic of the diagonal triangle touches the sides of LMN. Therefore if a cubic passes through A, B, C, A′, B′, C′ and touches the double lines NL, NM at Q, P, the tangent to the cubic at R (the ninth common point of intersection of the pencil of cubics through A, B, C, A′, B′, C′, P, Q) touches that inconic of the diagonal triangle which also touches the double lines NL, NM. Hence finally, since LM which is the tangent to the cubic at R (the point of contact of the cubic with its envelope) touches a fixed conic, this conic must be the same as the envelope of the cubic, which therefore establishes the truth of the reciprocal theorem.

Part II.

In this part, we shall give another proof of Morley's theorem based on a property relating to circular cubics given in Emch's *Projective Geometry*, page 206, *viz.* :

" *Considering a circular cubic in a Steinerian transformation, the asymptotes at the circular points intersect each other in a point D of the circle which corresponds to the line at infinity in the Steinerian transformation. The real asymptote cuts the same circle in a point C which with D determines a diameter of the circle. The points D, C are called the centre and principal point of the cubic.*"

Emch's proof of this lemma is *analytical*, but it may also be established by *pure geometry*. Let P be the real point at infinity on the cubic and X, Y, Z be the points of contact of the remaining tangents drawn from the principal point C to the cubic. With respect to the triangle XYZ, the circular cubic is self-isogonal, being the locus of isogonal conjugates on parallel rays through P. Tangents to the cubic at a pair of isogonal conjugate points Q, Q′ pass through the isogonal conjugate of the harmonic conjugate of P with respect to Q, Q′. Hence the centre D, usually called the double focus, is the isogonal conjugate of the point at infinity in a direction at right angles to P, and the principal point C is the isogonal conjugate of P ; so that C and D are diametrically opposite points on the

circum-circle of the triangle XYZ. The lines DX, DY, DZ are normals to the cubic.

If now, a circular cubic passes through the points A, B, C, A′, B′, C′ lying three by three on four lines, it passes through the focus S of the parabola touching these lines. For any cubic through these points, AA′, BB′, CC′, being the opposite vertices of the quadrilateral formed by the lines, form Steinerian couples, *i.e*, pairs of points tangents at which intersect on the cubic. The lines joining any point of the cubic to the points of a couple cut the cubic again in a couple (Emch, page 203). The system of co-axal circles through A and S, intersecting the circular cubic in four fixed points, cuts the cubic again on lines passing through a fixed point on the cubic. By considering the circles ABCS and AB′C′S, this fixed point is seen to be A′, the intersection of BC, B′C′. The radical axis AS and the line at infinity, forming a member of the co-axal system, cut the cubic again in a line through A, *i.e.* if P be the real point at infinity on the cubic, A′P and AS intersect on the cubic. Since A, A′ form a couple, P, S also form a couple. Hence we have the theorem :

The principal point of a circular cubic passing through six points lying three by three on four lines is the tangential of the focus of the parabola touching the lines.

If F is the double focus of the circular cubic, FS is normal to the cubic at S by Emch's theorem stated above. If F is a point on the directrix of the parabola touching the four lines, the tangent to the cubic at S being perpendicular to FS, is the polar of F with respect to the parabola. The polars of a given point with respect to the range of in-conics of a quadrilateral envelop the in-conic of the diagonal triangle of the quadrilateral touching the double lines of the pencil in involution determined at F by the range of in-conics. If F is a common point of intersection of the director circles of the range of the in-conics, the double lines coincide with the asymptotes of the cubic through F.

Hence if a variable cubic passing through six points lying three by three on four lines touches the double lines through any point F at G and H, we infer by projecting G and H into the circular points at infinity, that the tangent to the cubic at K, (the ninth point of intersection of the pencil of cubics through the six points and GFH) touches a fixed conic, *viz.*, that in-conic of the diagonal triangle of the quadrilateral formed by the four lines, which also touches the double lines. That the cubic must touch this conic at K can now be established as in Part I.

SHORT NOTES.

On the Integration of $\int d\theta / (a - b \cos\theta)^n$.

The integration of $\int_0^{\pi} \frac{d\theta}{(a-b\cos\theta)^n}$, when $a > b$ is reduced to the very neat form $\int_0^{\pi} (a^2-b^2)^{-n+\frac{1}{2}} (a+b\cos\phi)^{n-1}\, d\phi$ by the transformation*

$$(a - b\cos\theta)(a + b\cos\phi) = a^2 - b^2.$$

With this transformation, we have the differential relation

$$\frac{d\theta}{a-b\cos\theta} = \frac{d\phi}{\sqrt{a^2-b^2}};$$

and, in the class-room the amount of algebra needed to establish this, often prevents the student from duly appreciating the significance, of this method. The following geometrical proof, which is at once simple and direct, may be of some interest to the student.

Writing $(\alpha^2+\beta^2)$ for a and $2\alpha\beta$ for b, the condition $a > b$ is secured, and the problem is to show that with the transformation

$$(\alpha^2+\beta^2-2\alpha\beta\cos\theta)(\alpha^2+\beta^2+2\alpha\beta\cos\phi) = (\alpha^2-\beta^2)^2,$$

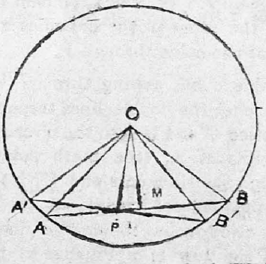
we have
$$\frac{d\theta}{\alpha^2+\beta^2-2\alpha\beta\cos\theta} = \frac{d\phi}{\alpha^2-\beta^2}.$$

In a circle of radius α, and centre O, introduce a chord APB, such that $OP = \beta$, $A\hat{O}P = \theta$, and $P\hat{O}B = \pi - \phi$.

Then $(\alpha^2+\beta^2-2\alpha\beta\cos\theta)^{\frac{1}{2}} (\alpha^2+\beta^2+2\alpha\beta\cos\phi)^{\frac{1}{2}}$

$$= AP \cdot PB = AM^2 - PM^2 = \alpha^2 - \beta^2$$

which is the transformation itself.



* Dr. Bromwich :—*A Short Table of Integrals*, p. 25.
Hardy :—*Cambridge Tract* No. 2, p. 44.
Hardy :—*Pure Mathematics*, p. 235.

Now taking a consecutive position of APB, we have in the limit

$$AA' = \alpha \,.\, d\theta, \; BB' = \alpha \,.\, d\phi$$

and

$$AA' : BB' = AP : PB'$$

$$= AP : PB \text{ in the limit.}$$

$$\therefore \quad \frac{d\theta}{AP} = \frac{d\phi}{PB}.$$

$$\therefore \quad \frac{d\theta}{\sqrt{(\alpha^2+\beta^2-2\alpha\beta\cos\theta)}} = \frac{d\phi}{\sqrt{(\alpha^2+\beta^2+2\alpha\beta\cos\phi)}};$$

i.e.

$$\frac{d\theta}{\alpha^2+\beta^2-2\alpha\beta\cos\theta} = \frac{d\phi}{\alpha^2-\beta^2}.$$

K. B. MADHAVA.

Note on the Equation a sec θ + b cosec $\theta = 1$.

The properties of this equation are usually obtained by substituting for sec θ and cosec θ the expressions $(1+t^2)/(1-t^2)$ and $(1+t^2)/2t$, where t stands for tan $\theta/2$; and noting the relations between the roots and co-efficients of the resulting biquadratic equation.

The following method may be suggested :—

Consider the result

$$(l\cos\theta + m\sin\theta + 1)\left(\frac{\cos\theta}{l} + \frac{\sin\theta}{m} - 1\right)$$

$$= \cos^2\theta + \sin^2\theta + \cos\theta\sin\theta\left(\frac{l}{m} + \frac{m}{l}\right)$$

$$+ \cos\theta\left(\frac{1}{l} - l\right) + \sin\theta\left(\frac{1}{m} - m\right) - 1$$

$$= \cos\theta\sin\theta\left(\frac{l}{m} + \frac{m}{l}\right) - \cos\theta\left(l - \frac{1}{l}\right)$$

$$- \sin\theta\left(m - \frac{1}{m}\right).$$

The equation

$$a\sec\theta + b\,\mathrm{cosec}\,\theta - 1 = 0, \qquad \ldots \qquad \ldots \qquad (1)$$

which is the same as

$$\cos\theta\sin\theta - b\cos\theta - a\sin\theta = 0,$$

may therefore be written

$$(l\cos\theta + m\sin\theta + 1)\left(\frac{\cos\theta}{l} + \frac{\sin\theta}{m} - 1\right) = 0,$$

where $a = \dfrac{(m^2-1)l}{l^2+m^2}$, $b = \dfrac{(l^2-1)m}{l^2+m^2}$.

The roots of the proposed trigonometrical equation are thus separable into those of

$$l \cos \theta + m \sin \theta + 1 = 0, \ldots \quad \ldots \quad \ldots \quad (2)$$

and

$$\frac{\cos \theta}{l} + \frac{\sin \theta}{m} - 1 = 0. \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Now, if α, β be the distinct roots of (2) and γ, δ those of (3), we know

$$l : m : 1 = \cos \frac{\alpha+\beta}{2} : \sin \frac{\alpha+\beta}{2} : \cos \frac{\alpha-\beta}{2};$$

and

$$\frac{1}{l} : \frac{1}{m} : -1 = \cos \frac{\gamma+\delta}{2} : \sin \frac{\gamma+\delta}{2} : \cos \frac{\gamma-\delta}{2}.$$

Hence $\cos \frac{\alpha+\beta}{2} \cos \frac{\gamma+\delta}{2} - \sin \frac{\alpha+\beta}{2} \sin \frac{\gamma+\delta}{2} = 0,$

i.e.,

$$\cos \frac{\alpha+\beta+\gamma+\delta}{2} = 0,$$

or

$$\alpha + \beta + \gamma + \delta = (2n+1)\pi \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Also, from (3)

$$\frac{\cos \gamma}{l} + \frac{\sin \gamma}{m} + 1 = 0,$$

whence substituting for l and m

$$\frac{\cos \gamma \cos \frac{\alpha-\beta}{2}}{\cos \frac{\alpha+\beta}{2}} + \frac{\sin \gamma \cos \frac{\alpha-\beta}{2}}{\sin \frac{\alpha+\beta}{2}} + 1 = 0;$$

i.e., $\cos \gamma (\sin \alpha + \sin \beta) + \sin \gamma (\cos \alpha + \cos \beta) + \sin (\alpha + \beta) = 0,$

i.e., $\sin (\alpha+\beta) + \sin (\beta+\gamma) + \sin (\gamma+\alpha) = 0. \ldots \quad \ldots \quad \ldots \quad (5)$

M. T. NARANIENGAR.

SOLUTIONS.

Question 624.

(A. N. RAGHAVACHAR):—Two parabolas of latera recta $4a$, $4b$ have a common vertex, and their axes are inclined at an angle α. If they again intersect at an angle $(\alpha - \beta)$ and if the distance between their foci is c, prove that

$$\frac{c^2}{ab \cos \alpha} = \frac{9t^2 s\,(3ts^2 - 4t + 8s)}{4(t-2s)^2\,(2t-s)},$$

where t, s stand for $\tan \alpha$, $\tan \beta$:

Additional Solution by Eric H. Neville.

Take for axes of co-ordinates the axes of the two parabolas. Then the equations of the curves are

$$\left.\begin{aligned} y^2 \sin^2 \alpha &= 4a\,(x + y \cos \alpha), \\ x^2 \sin^2 \alpha &= 4b\,(y + x \cos \alpha). \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

At the common point, let the first tangent make an angle θ with the x-axis, the second an angle ψ with the y-axis; by hypothesis

$$\theta + \psi = \beta \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

At the common point, the slope of the first tangent is $\sin\theta \div \sin(\alpha - \theta)$; hence if (p, q) is the common point,

$$q \sin^2\alpha \sin\theta = 2a\{\sin(\alpha - \theta) + \cos\alpha \sin\theta\},$$

and therefore

$$q \sin\alpha \sin\theta = 2a \cos\theta ; \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

and by substitution in (1)

$$2p \sin\theta = q(\sin\alpha \cos\theta - 2\cos\alpha \sin\theta). \dots \quad \dots \quad (4)$$

Similarly

$$p \sin\alpha \sin\psi = 2b \cos\psi, \dots \quad \dots \quad \dots \quad \dots \quad (5)$$

$$2q \sin\psi = p(\sin\alpha \cos\psi - 2\cos\alpha \sin\psi) ; \dots \quad \dots \quad (6)$$

and by multiplication of (4) and (6)

$$\sin\alpha(\cos\theta\cos\psi - 4\sin\theta\sin\psi) = 2\cos\alpha\sin\beta \quad \dots \quad (7)$$

from (2). Thus if we write

$$C = \cos\theta\cos\psi, \; S = \sin\theta\sin\psi,$$

we have from (2) and (7)

$$C - S = \cos\beta,$$

$$t(C - 4S) = 2\sin\beta,$$

and therefore

$$3tC = 4t\cos\beta - 2\sin\beta = 2(2t - s)\cos\beta \quad \dots \quad (8)$$

$$3tS = t\cos\beta - 2\sin\beta = (t - 2s)\cos\beta \quad \dots \quad \dots \quad (9)$$

Again from (3), (5), (6),

$$\frac{a}{b} = \frac{(\sin\alpha\cos\psi - 2\cos\alpha\sin\psi)\sin\theta\cos\psi}{2\cos\theta\sin^2\psi},$$

that is,

$$\frac{2CS^2}{\cos\alpha} \cdot \frac{a}{b} = (t\sin\theta\cos\psi - 2S)\sin^2\theta\cos^2\psi ;$$

similarly

$$\frac{2CS^2}{\cos\alpha} \cdot \frac{b}{a} = (t\sin\psi\cos\theta - 2S)\sin^2\psi\cos^2\theta.$$

Hence, from (2),

$$\frac{2CS^2}{\cos\alpha}\left(\frac{a}{b} + \frac{b}{a}\right) = t\sin\beta(\sin^2\theta\cos^2\psi + \sin^2\psi\cos^2\theta - CS)$$

$$- 2S(\sin^2\theta\cos^2\psi + \sin^2\psi\cos^2\theta)$$

$$= (t\sin\beta - 2S)(\sin^2\beta - 2CS) - tCS\sin\beta,$$

whence

$$\frac{2CS^2}{\cos\alpha}\left(\frac{a}{b} + \frac{b}{a} - 2\cos\alpha\right) = \{t\sin^2\beta - S(2\sin\beta + 3tC)\}\sin\beta$$

$$= (t\sin^2\beta - 4tS\cos\beta)\sin\beta,$$

from (8), that is,

$$\frac{c^3}{ab\cos\alpha} = \frac{s\{3ts^2 - 4(t-2s)\}\cos^2\beta}{60S^2}$$

and (8), (9) shew that this is the required result.

It is worth while to add that there is not in this solution any question of *familiarity* with oblique axes; (1) is obvious enough, and (3), in the form

$$\cot\theta = q\sin\alpha / 2a,$$

expresses a well-known property.

Question 854.

(MARTYN M. THOMAS, M A.)—If ρ and σ be the radii of curvature at corresponding points of a curve and its α-evolute of the β-evolute, where α, β are constant, shew that

$$\frac{\sigma}{\rho} = \left(\frac{d\rho}{ds}\cos\alpha + \sin\alpha\right)\left(\frac{d\rho}{ds}\cos\beta + \sin\beta\right) + \rho\frac{d^2\rho}{ds^2}\cos\alpha\cos\beta.$$

Additional Solution and Remarks by Eric H. Neville.

A reader with a healthy distrust of infinitesimals may be interested to compare the following solution with that on p. 197 of Vol. XI of the Journal.

Let O_1, O_2, O_3 be corresponding current points of the original curve, of the β-evolute, and of the α-evolute of the latter; let the tangent and normal at O_k to the curve described by O_k be O_kT_k and O_kN_k. With respect to any variable t let the point O_k have speed w_k in its curve. The three frames $O_1T_1N_1$, $O_2T_2N_2$, $O_3T_3N_3$, have the same spin with respect to t, since the angles α, β are constants; denote this spin by θ. Then the radii of curvature of the curves are w_1/θ, w_2/θ, w_3/θ. Denote the lengths O_1O_2, O_2O_3 by p, q.

The frame with origin O_1 and axes parallel to O_2T_2 and O_2N_2 is a frame with translations $w_1\sin\beta$, $-w_1\cos\beta$ and spin θ; hence the point whose co-ordinates in this frame are p, 0 has a velocity whose components are $(dp/dt) + w_1\sin\beta$, $-w_1\cos\beta + \theta p$. Since this is the point O_2, whose velocity by hypothesis is w_2 in O_2T_2,

$$-w_1\cos\beta + \theta p = 0, \qquad \text{which determines } p,$$

and

$$w_2 = (dp/dt) + w_1\sin\beta.$$

Repeating the argument,

$$-w_2\cos\alpha + \theta q = 0,$$
$$w_3 = (dq/dt) + w_2\sin\alpha,$$

and substituting s for t and writing

$$w_1 = 1,\ \theta = 1/\rho,\ w_3 = \sigma/\rho.$$

we have

$$p = \rho \cos \beta,$$

$$w_2 = \frac{d\rho}{ds} \cos \beta + \sin \beta,$$

$$q = \rho\, w_2 \cos \alpha,$$

$$w_3 = \rho \frac{dw_2}{dt} \cos \alpha + w_2 \left(\frac{d\rho}{ds} \cos \alpha + \sin \alpha \right),$$

and the last formula is the result required, with the factors already in evidence.

The calculation is much the same as in the solution given; it is the point of view that is different, and I venture to think, safer.

Question 857.

(S. RANGANATHAN):—Show that the following equation represents a line pair:—

$$f(x)X^2 + 2f(xy)XY + f(y)Y^2 + 2f(x, z)X + 2f(y, z)Y + f(z) = 0$$

where X and Y are the current cartesian co-ordinates of a point, and $f(x) = (a, h, b)(x, x_1, x_2)^2$, etc.; and $f(x, y) = (a, h, b)(x_1\ x_2)(y_1\ y_2)$, etc.

Additional Solution by H. Br.

Writing

$$\xi = x_1 X + y_1 Y + z_1$$
$$\eta = x_2 X + y_2 Y + z_2$$

it is easily seen, by direct expansion or otherwise, that the given quadratic expression is equal to $f(\xi, \eta)$, *i.e.*, to

$$a\xi^2 + 2h\xi\eta + b\eta^2,$$

and is therefore a product of two linear factors.

Hence the equation represents a line pair.

Question 941.

(BALAK RAM):—Solve

$$x! = (x - y)!\,(y)!\ P.\ Q.\ R.$$

where x and y may be any integers satisfying $3 < y \leq \frac{x}{2}$ and P, Q, R are primes. [Suggested by Q. 817].

Solution by K. B. Madhava.

At the outset it is obvious that there are an infinite number of solutions for x if $y = 1$, 2 or 3. For example in Q. 817, $y = 3$ and there is an additional condition given with respect to the sum of the primes.

Hence y should be greater than 3. If we should take also the upper limit of y, we get that $x = 2y + z$, where z should be positive and may be zero.

Then the problem is to determine y and z such that

$$\frac{(2y + z)(2y - 1 + z)\ldots\ldots(y + 1 + z)}{y\,!} = \text{product of 3 primes} = \text{P. Q. R.}$$

Consequently,

$$(2y + z)\,(2y - 1 + z)(2y - 2 + z) > \text{P. Q. R.}$$

since each of P, Q, R should divide at least one number in the numerator and the three terms on the left of the inequality are the three highest in the numerator.

$$\text{Hence } \frac{(2y - 3 + z)\,(2y - 4 + z)\ldots\ldots(y + 2 + z)(y + 1 + z)}{y\,!} < 1.$$

But z is never negative.

Hence *á fortiori*

$$(2y - 3)\,(2y - 4)\ldots\ldots(y + 2)(y + 1) < y\,!$$

Solving this inequality tentatively we see that y cannot be greater than 6. Therefore the values $y = 4$, $y = 5$, $y = 6$, should be examined.

If $y = 6$,

$$(9 + z)\,(8 + z)\,(7 + z) < 6\,! < 720.$$

Hence $z = 0$.

$\therefore\ x = 12$.

Now $12c_6$ cannot be exhibited as the product of any three primes. Therefore no solution exists in this case.

Take next $y = 5$;

then $(7 + z)\,(6 + z) < 5\,! < 120.$

Hence $z < 5$,

i.e., $x = 14, 13, 12, 11, 10$.

None of these again satisfy the conditions of the problem.

Finally, with $y = 4$, $z < 19$

hence x may be any integer between 4 and 23. And xc_4 will be a product of three primes only in the following cases :—

$$\left.\begin{aligned} 8c_4 &= 2\,.\,5\,.\,7 \\ 13c_4 &= 5\,.\,11\,.\,13 \\ 14c_4 &= 7\,.\,11\,.\,13 \end{aligned}\right\}.$$ Hence these are the only three solutions.

Question 942.

(Hemraj) :—Show that if p be prime,

$$\left[\left(\Sigma_1^{p-1} \frac{1}{n} \right) \div \frac{p^3}{p!} \right] \text{ is an integer.}$$

Solution by R. J. Pocock.

$$\left[\Sigma_1^{p-1} \left(\frac{1}{n} \right) \right] \div \frac{p^3}{p!}$$

$$= \frac{1}{p^2} \left\{ \frac{\overline{p-1}.!}{1} + \frac{\overline{p-1}!}{2} + \frac{\overline{p-\iota}!}{3} + \ldots + \frac{\overline{p-1}.!}{p-1} \right\}.$$

Taking the first and last, second and last but one of the terms we have the right side

$$= \frac{1}{p} \left\{ \frac{\overline{p-1}!}{1 \, . \, \overline{p-1}} + \frac{\overline{p-1}!}{2 \, . \, \overline{p-2}} + \frac{\overline{p-\iota}!}{3 \, . \, \overline{p-3}} + \ldots \right\} \qquad \ldots \quad (1)$$

Now by Lagrange's Theorem the sum of the products $(p-2)$ at a time of the quantities 1, 2, 3 …… $(p-1)$ is divisible by p.

Consider all such products $\frac{\overline{p-\iota}!}{r \, . \, s}$ where $r = s$ and $r + s = p$ and take them in pairs thus $\frac{\overline{p-1}!}{r \, . \, s} + \frac{\overline{p-\iota}!}{(p-r) \, . \, s} = \frac{\overline{p-1}! \, p \, .}{(p-r) \, . \, r \, . \, s}$ this is an integer since $r = s$, $p - r = s$. Hence the sum of all such products $p-2$ at a time exclusive of those inside the bracket in (1) above is divisible by p. Hence the sum of the remaining products, *viz.*, those inside the bracket in (1) is divisible by p. Hence the theorem.

Question 931.

(H. Br.) :—Show that

$$x^2 = y^2 + z^2$$
$$t^2 = y^2 + 4z^2$$

have no solutions in non-zero integers.

Solution by the Proposer.

Take y, z to be prime to each other and z even, as we may do without loss of generality.

$$\left. \begin{array}{l} y = l^2 - m^2 \\ z = 2lm \end{array} \right\} \text{ and also } \left. \begin{array}{l} y = \mathrm{L}^2 - \mathrm{M}^2 \\ 2z = 2\mathrm{LM} \end{array} \right\}$$

We may take y to be + or —, and ensure that m and M are even. [It is impossible that one should be odd and the other even].

From $2lm = 2\mathrm{LM}$, we conclude

$$\left. \begin{array}{l} l = \alpha\beta \\ m = \gamma\delta \end{array} \right\} \left. \begin{array}{l} \mathrm{L} = \alpha\gamma \\ \mathrm{M} = 2\beta\delta \end{array} \right\}$$

δ being even, and α, β, γ odd.

Also as l, m are co-primes, and L, M are co-primes, α, β, γ, δ are all prime to each other.

Substituting for l, m, L, M in the two values of y,

$$\alpha^2\beta^2 - \gamma^2\delta^2 = \alpha^2\gamma^2 - 4\beta^2\delta^2$$

or $$\beta^2(\alpha^2 + 4\delta^2) = \gamma^2(\alpha^2 + \delta^2).$$

$$\therefore \quad \frac{\alpha^2 + 4\delta^2}{\gamma^2} = \frac{\alpha^2 + \delta^2}{\beta^2} = v, \text{ an integer };$$

v is a positive common factor of $\alpha^2 + \delta^2$ and $\alpha^2 + 4\delta^2$, and is therefore either 1 or 3 as α and δ are co-primes. But if $\alpha^2 + \delta^2$ is divisible by 3, both α, δ are divisible by 3, and are therefore not co-prime.

Hence $v = 1$.

$$\left.\begin{aligned}\beta^2 &= \alpha^2 + \delta^2 \\ \gamma^2 &= \alpha^2 + 4\delta^2\end{aligned}\right\}$$

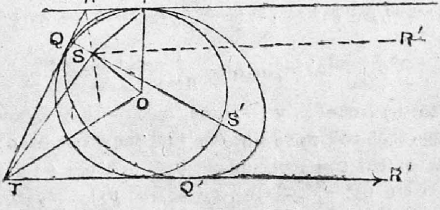
a relation of the original form, with $\alpha < y$. Proceeding in this manner we arrive at $\beta_0{}^2 = 1 + \delta_0{}^2$ which has no non-zero solutions. Hence the problem.

Question 1036.

(A. C. L. Wilkinson) :—A variable circle touches an ellipse at a fixed point. Show that the locus of the point of intersection of the other two common tangents of the circle and ellipse is the confocal hyperbola passing through the fixed point.

Solution (1) by K. J. Sanjana and (2) by G. N. G. Dathar.

(1) Let the circle whose centre is O touch the given ellipse at the fixed point P; let RPR′ be the tangent at P, and let the other common tangents of the circle and ellipse meet in T and touch the ellipse at Q, Q′, forming with the fixed tangent the triangle TRR′. Let S and S′ be the foci, produce RS to r.



Since from the ⊙ ∠OTQ = ∠OTQ′,
and from the ellipse ∠STQ = ∠S′TQ′,

$\therefore \angle OTS = \angle OTS'$: thus OT is the tangent at T to the confocal hyperbola through T. So also OR, OR′ are tangents to the confocal hyperbolas (or ellipses) through R, R′ at these points.

Now the locus of poles of the fixed line RPR′ for the conic system whose foci are S, S′, is a st. line perpendicular to RR′ ; and ∵ the pole of RR for the given ellipse is P itself, ∴ the locus of poles is the normal at P, *i.e.*, the line OP. And because for two points R and R′ on the fixed line, one tangent goes through O on the line which is the locus, it follows that O is the pole of RR′ itself for another conic, and therefore that R and R′ lie on the same confocal.

Hence, as the figure is drawn, ∠s RSO, R′SO are supplements, and ∴ R′SO = OS*r*.

$$\text{But } \angle PSR' = \tfrac{1}{2}PSQ' = \tfrac{1}{2}(2\pi - PSQ - QSQ')$$
$$= \pi - RSQ - QST = \pi - RST = TSr.$$

Therefore $\angle PSR' + R'SO = \angle TSr + rSO$, *i.e.*. $\angle PSO = \angle TSO$;

and ∵ S is a common focus, it follows that OT and OP are tangents to the same confocal.

(2) Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0$ and the fixed point $(a \cos \theta, b \sin \theta)$.

A circle touching the ellipse at the fixed point will be of the form $[x - (a + b\lambda) \cos \theta]^2 + [y - (b + a\lambda) \sin \theta]^2 = \lambda^2(a^2 \sin^2 \theta + b^2 \cos^2 \theta)$.

Moving the origin to the fixed point, these equations become

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2x}{a} \cos \theta + \frac{2y}{b} \sin \theta = 0 \qquad \ldots \quad \text{(i)}$$

and
$$x^2 + y^2 - 2b\lambda x \cos \theta - 2a\lambda y \sin \theta = 0 \qquad \ldots \quad \text{(ii)}$$

Tangents to (i) from (x', y') will be

$$\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2x \cos \theta}{a} + \frac{2y \sin \theta}{b}\right] \left[\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{2x' \cos \theta}{a} + \frac{2y' \sin \theta}{b}\right]$$
$$- \left[\frac{x}{a}\left(\frac{x'}{a} + \cos \theta\right) + \frac{y}{b}\left(\frac{y'}{b} + \sin \theta\right) + \frac{x' \cos \theta}{a} + \frac{y' \sin \theta}{b}\right]^2 = 0$$

or
$$\frac{x^2}{a^2b^2}[(y' + b \sin \theta)^2 - b^2] - \frac{2xy}{a^2b^2}(x' + a \cos \theta)(y' + b \sin \theta)$$
$$+ \frac{y^2}{a^2b^2}[(x' + a \cos \theta)^2 - a^2] + \ldots\ldots - \frac{1}{a^2b^2}[bx' \cos \theta + ay' \sin \theta]^2 = 0.$$

Tangents to (ii) from (x', y') will be

$$[x^2 + y^2 - 2b\lambda x \cos \theta - 2a\lambda y \sin \theta][x'^2 + y'^2 - 2b\lambda x' \cos \theta - 2a\lambda y' \sin \theta]$$
$$-[x(x' - b\lambda \cos \theta) + y(y' - ay \sin \theta) - (b\lambda x' \cos \theta + a\lambda y' \sin \theta)]^2 = 0$$

or
$$x^2[(y' - a\lambda \sin \theta)^2 - \lambda^2(a^2 \sin^2 \theta) + b^2 \cos^2 \theta)] - 2xy(x' - b\lambda \cos \theta)$$
$$\times (y' - a\lambda \sin \theta) + y^2[(x' - b\lambda \cos \theta)^2 - \lambda^2(a^2 \sin^2 \theta + b^2 \cos^2 \theta)]$$
$$- \ldots\ldots \lambda^2[ax' \cos \theta + ay' \sin \theta]^2 = 0.$$

If (x', y') be the intersection of the other two common tangents to (i) and (ii), these two equations will be identical.

$$\therefore \frac{y'^2 - 2a\lambda y' \sin\theta - b^2\lambda^2 \cos^2\theta}{(y' + b\sin\theta)^2 - b^2} \ldots\ldots\ldots$$

$$= \frac{x'^2 - 2b\lambda x' \cos\theta - a^2\lambda^2 \sin^2\theta}{x'^2 + 2ax'\cos\theta - a^2\sin^2\theta} = \ldots\ldots = \lambda^2$$

$$\therefore \quad \lambda^2 x'(x' + 2a'\cos\theta) + 2b\lambda x'\cos\theta - x'^2 = 0$$

or $$\lambda^2(x' + 2a\cos\theta) + 2b\lambda\cos\theta - x' = 0$$

and $$\lambda^2(y' + 2b\sin\theta) + 2a\lambda\sin\theta - y' = 0.$$

Eliminating λ,

$$\frac{\lambda^2}{ax'\sin\theta - by'\cos\theta} = \frac{\lambda}{(ay'\cos\theta - bx'\sin\theta)}$$

$$= \frac{1}{ax'\sin\theta - by'\cos\theta + 2(a^2 - b^2)\,\text{cof}\,\theta\sin\theta}$$

Moving the origin back to the centre, we have

$$\frac{\lambda^2}{(ax'\sin\theta - by'\cos\theta) - (a^2 - b^2)\sin\theta\cos\theta} = \frac{\lambda}{ay'\cos\theta - bx'\sin\theta}$$

$$= \frac{1}{(ax'\sin\theta - by'\cos\theta) + (a^2 - b^2)\cos\theta\sin\theta}$$

Hence the required locus is

$$(ax'\sin\theta - by'\cos\theta)^2\,(a^2 - b^2)^2\sin^2\theta\cos^2\theta = (ay'\cos\theta - bx'\sin\theta)^2$$

or, dropping the dashes,

$$(a^2 - b^2)x^2\sin^2\theta - (a^2 - b^2)y^2\cos^2\theta = (a^2 - b^2)^2\sin^2\theta\cos^2\theta,$$

or $$\frac{x^2}{(a^2 - b^2)\cos^2\theta} - \frac{y^2}{(a^2 - b^2)\sin^2\theta} = 1$$

or $$\frac{x^2}{a^2 - (a^2\sin^2\theta + b^2\cos^2\theta)} + \frac{y^2}{b^2 - (a^2\sin^2\theta + b^2\cos^2\theta)} = 1,$$

which is the confocal hyperbola through the fixed point.

Question 1038.

(A. C. L. WILKINSON) :—The equation of a conic referred to tangential areal co-ordinates is

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0;$$

show that the magnitudes of its semi-axes are given by the equation

$$I^3x^4 + 4IJR^2x^2 + 16R^4\,\Delta\sin^2 A\sin^2 B\sin^2 C = 0,$$

where R is the radius of the circum-circle of the triangle of reference,

and $$I = a + b + c + 2f + 2g + 2h$$

$$J = A\sin^2 A + B\sin^2 B + C\sin^2 C - 2F\cos A\sin B\sin C - 2G\cos B\sin C\sin A - 2H\cos C\sin A\sin B,$$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, A = \frac{d\Delta}{da}, F = \tfrac{1}{2}\frac{d\Delta}{df}, \text{ etc.}$$

Additional Solution by E. H. Neville.

To get the full benefit from questions of this kind, solvers should aim at a process that goes to the heart of the matter.

In areal co-ordinates, the line at infinity has co-ordinates 1, 1, 1 and therefore the centre of the conic

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \quad \ldots \quad (1)$$

has the equation

$$pl + qm + rn = 0$$

where

$$p = a + h + g, q = h + b + f, r = g + f + c; \quad \ldots \quad (2)$$

in other words, p, q, r are the co-ordinates of the centre.

For *any* values of p, q, r, and l, m, n, the distance from the point p, q, r to the line l, m, n is proportional to

$$\frac{pl + qm + rn}{(p+q+r)\sqrt{(l^2\sin^2 A + m^2\sin^2 B + n^2\sin^2 C - \Sigma 2mn\sin B\sin C\cos A)}}$$

and is therefore actually equal to this fraction multiplied by

$$2R\sin A\sin B\sin C,$$

since the distance from A, which is the point (1, 0, 0) to BC, which is the line (1, 0, 0) is 2R sin B sin C.

The question, in elementary terms, becomes:

"If l, m, n are subject to (1) and to

$$Ll^2 + Mm^2 + Nn^2 + 2Pmn + 2Qnl + 2Rlm = 1 \quad \ldots \quad (3)$$

where

$$L = \sin^2 A, M = \sin^2 B, N = \sin^2 C$$

$$P = -\sin B\sin C\cos A, Q = -\sin C\sin A\cos B, R = -\sin A\sin B\cos C,$$

what is the equation satisfied by stationary values of

$$k(pl + qm + rn),$$

where p,q,r have the values given in (2) and $k = 2R\sin A\sin B\sin C/I$?" (4)

The answer is that the equation results from eliminating l, m, n between the two equations of condition (1), (3), the equation

$$pl + qm + rn = x/k, \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

and the equation

$$\begin{vmatrix} Ll + Rm + Qn & al + hm + gn & p \\ Rl + Mm + Pn & hl + bm + fn & q \\ Ql + Pm + Nn & gl + fm + cn & r \end{vmatrix} = 0,$$

that is, from eliminating l, m, n, λ, μ between (1), (3), (5) and

$$\left.\begin{aligned} \lambda(Ll + Rm + Qn) + \mu(al + hm + gn) &= p, \quad (6\cdot1) \\ \lambda(Rl + Mm + Pn) + \mu(hl + bm + fn) &= q, \quad (6\cdot2) \\ \lambda(Ql + Pm + Nn) + \mu(gl + fm + cn) &= r, \quad (6\cdot3) \end{aligned}\right\} \quad \ldots \quad (6)$$

Since identically

$$L + R + Q = 0,\ R + M + P = 0,\ Q + P + N = 0, \quad \ldots \quad (7)$$

addition of (6·1), (6·2), (6·3) gives

$$\mu(x/K) = I,$$

while multiplying (6·1), (6·2), (6·3) by l, m, n and adding gives

$$\lambda = x/K.$$

Hence (6) can be replaced by

$$\left.\begin{aligned} l(Lx^2 + aIK^2) + m(Rx^2 + hIK^2) + n(Qx^2 + gIK^2) &= pKx, \\ l(Rx^2 + hIK^2) + m(Mx^2 + bIK^2) + n(Px^2 + fIK^2) &= qKx, \\ l(Qx^2 + gIK^2) + m(Px^2 + fIK^2) + n(Nx^2 + cIK^2) &= rKx, \end{aligned}\right\} (8)$$

and from (8) and (5) an equation for x comes at once.

An immediate simplification is suggested by (7), which with (2) gives three *identities* of which the first is

$$(Lx^2 + aIK^2) + (Rx^2 + hIK^2) + (Qx^2 + gIK^2) = pIK^2,$$

shewing that (8) can be replaced by

$$\{ l - (IK/x) \} (Lx^2 + aIK^2) + \{ m - (IK/x) \} (Rx^2 + hIK^2) + \{ x - (IK/x) \} (Qx^2 + gIK^2) = 0,$$

and two similar equations, and since these are *homogeneous* in

$l - (IK/x, m - (IK/x), x - (IK/x)$, the eliminant takes the form

$$\begin{vmatrix} Lx^2 + nIK^2 & Rx^2 + hIK^2 & Qx^2 + gIK^2 \\ Rx^2 + hIK^2 & Mx^2 + bIK^2 & Px^2 + fIK^2 \\ Qx^2 + gIK^2 & Px^2 + fIK^2 & Nx^2 + cIK^2 \end{vmatrix} = 0.$$

When this is expanded, the co-efficient of x^6 is zero by (7), the co-efficient of x^2 is I^2JK^4, where J has the value given in the question, and the constant term is $I^3 \Delta K^6$. As for the co-efficient of x^4, since identically

$$MN - P^2 = QR - LP = \sin^2 A \sin^2 B \sin^2 C,$$

this is $IK^2(a + b + c + 2f + 2g + 2h) \sin^2 A \sin^2 B \sin^2 C$,

that is, $I^2K^2 \sin^2 A \sin^2 B \sin^2 C$.

The equation is therefore

$$x^4 \sin^2 A \sin^2 B \sin^2 C + JK^2x^2 + I\Delta K^4 = 0,$$

which requires nothing but substitution of the value of K from (4) to take the form required.

[*N.B.* The value of J is quite correct, but Mr. Sanjana doubts its correctness on p. 189 of the October Journal.]

Questions 1043 and 1044.

(V. RAMASWAMI AIYAR, M.A.) :—On the sides BC, CA, AB of a Δ ABC, pairs of points x, x', y, y', z, z' are taken such that $xx' = $ BC, $yy' = $ CA, $zz' = $ AB and $\perp$rs are erected at these points to the sides. If the $\perp$rs at (y, z'), (z, x'), (x, y') intersect at P, Q, R and those at (y', z), (z', x), (x', y) intersect at P′, Q′, R′, show that P, Q, R, P′, Q′, R′, are concyclic.

(V. RAMASWAMI AIYAR, M.A.) :—Given three straight lines as well as their orthopoles : determine the circum-circle as well as the nine points cricle, of the triangle of reference. Show also how the triangle of reference could be determined.

Solution of Q. 1043 by V. V. S. Narayan.

Let PR′, RQ′ intersect at c and let RK, R′K′ be $\perp$rs to them from R, R′.

Then, by hypothesis, RK $= yy' =$ AC, and R′K′ $= xx' =$ BC.

Since $\frac{1}{2}(c\text{R}, \text{R}'\text{K}')$ and $\frac{1}{2}(c\text{R}', \text{RK})$ are each = area of Δ cRR′, $c\text{R}/c\text{R}' = \text{RK}/\text{R}'\text{K}' = \text{AC}/\text{BC}$. But Δs cPQ′ and ABC are similar since their sides are $\perp$r to one another, so that $c\text{P}/c\text{Q}' = \text{AC}/\text{BC}$. Therefore $c\text{R}/c\text{R}' = c\text{P}/c\text{Q}'$, *i.e.*, $c\text{P} \,.\, c\text{R}' = c\text{R} \,.\, c\text{Q}'$.

Hence, R, R′, Q′, P are cyclic.

Similarly it can be shown that R′, P′ are concyclic with R, R′, Q′, P.

Note on the above Question by V. V. S. Narayan.

1. Let $a'b'c'$ and abc be the Δs formed by (RP′, PQ′, QR′) and (R′P, P′Q, Q′R) respectively. Then since RR′ is anti-parallel to $a'b'$, and cc' bisects it being the other diagonal of the parallelogram Rc' Rc' Rc' is a symmedian of the Δ $a'b'c'$. Similarly, aa', bb' are also symmedians. Hence the centre of perspective of the Δs abc, $a'b'c'$ is the symmedian point of the Δ $a'b'c'$.

2. The Δs abc, $a'b'c'$, ABC are similar.

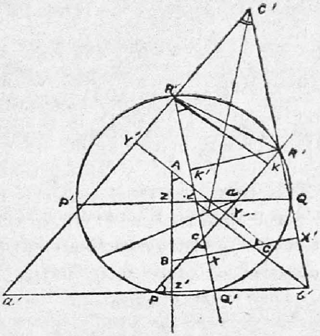
3. The circle PP′ QQ′ RR′ is a TUCKER circle of the Δ $a'b'c'$ and its centre lies midway between the circumcentres of Δs abc, $a'b'c'$. The symmedian point of Δ $a'b'c'$ is collinear with these three centres.

4. Since LR′RQ′ = LR′PQ′ = LA of Δ ABC and R′K′ is $\perp$r to RQ′ and is = BC, RR′ = R′K′/sin R′RK′ = BC/sin A = 2R = circumdiameter of Δ ABC. Similarly PP′ = QQ′ = RR′ = circumdiameter of the Δ ABC.

Solutions similar to the above by R. S Mahajan, V. M. Gaitonde and M. K. Kewalramani.

Solution of Questions 1043 *and* 1044 *by M. Bhimasena Rao.*

It is known that the orthopole of a line given in direction lies on the pedal line of the triangle of reference, which is perpendicular to the given direction and that the distance of the orthopole from the line is constant as the line varies—*vide* Orthopole Theorems, Gallatley's *Modern Geometry*. From the given orthopoles draw perpendiculars to the given lines forming a triangle ABC and let the distances of the orthopoles from their corresponding lines be p, q, r. On BC, take $Bx = Cx' = p$ and draw lines through x and x' at right angles to BC. Similarly cut off lengths on CA, AB, equal to q, r respectively and erect perpendiculars to these sides at the points of section. Let the perpendiculars to the sides of ABC intersect in the points P, Q, R, P', Q', R' as in the figure.



It is now evident that C is the orthopole of x'R' and also that of yR'. Therefore the point R' lies on the circum-circle of the triangle of reference, which for a similar reason passes through P' and Q'. Since BC is the pedal line through the point C and perpendicular to x'R', its pole must lie on yR', the orthopole of which is also C. Considering the point B, it follows by similar reasoning that the pole of BC lies on z'Q'. Therefore P, the point of intersection of yR' and z'Q' is the pole of BC, and lies on the circum-circle of the triangle of reference. Hence the six points PQR P'Q'R' are concyclic. This is the result in Q. 1043.

For an independant proof of Q. 1043, we observe that the projections of PP' on CA, AB being equal respectively to CA, AB,—PP' must be equal and parallel to the circumdiameter of ABC through A. It follows that PP' = QQ' = RR'. QR' is parallel to Q'R and QQ' = RR'. Therefore

the points QQ′ RR′ are concyclic. RR′ makes with BC the angle 90—A

$$\text{the angle } RR'x = 180 - A,$$
$$\text{angle } PR'x = C.$$
$$\therefore \text{ angle } RR'P = B.$$
$$\text{Angle } RQ'P = 180 - B.$$

Therefore P lies on the circle QQ′ RR′.

Similarly P′ lies on this circle.

Since ABC is the triangle formed by the pedal lines of P, Q, R, the reflexion of the orthocentre of PQR in the circumcentre of ABC is the orthocentre of the triangle of reference. *Vide* my extension of Mr. S. Narayanan's question 178, *vide* Q. 291, page 128, Vol. III, J. I. M. S.

The circum-circle of the triangle of reference and its orthocentre being known, the nine point circle is known. The sides of the triangle of reference are the common tangents of the parabolas with foci at P, Q, R and vertex tangents BC, CA, AB.

Question 1051.

(A. C. L. Wilkinson) :—If ABCD is a cyclic quadrilateral, the points A, B, C, D occurring in this cyclic order, the sum of the reciprocals of the perpendiculars from A, C on their Simson lines with respect to the triangles BCD, ABD, respectively is equal to the sum of the reciprocals of the perpendiculars from B, D, on their Simson lines with respect to the triangles ACD, ABC respectively.

Solutions by V. M. Gaitonde, R. S. Mahajan and M. K. Kewalramani.

If L, M be the feet of the perpendiculars from A on DC, DB respectively and if p_1, p_2, p_3, p_4 be the perpendiculars from A, B, C, D on their respective Simson lines, we have

$$p_1 = AL \sin ALM = AD \sin ADC \sin ADB = 4R^2 . AD . AB . AC$$

where R is the radius of the circle ABCD.

$$\text{Similarly } p_2 = 4R^2 . BA . BC . BD;$$
$$p_3 = 4R^2 . CD . CA . CB;$$
$$p_4 = 4R^2 . DA . DB . DC.$$

$$\text{Hence } \frac{1}{p_1} + \frac{1}{p_3} = \frac{1}{4R^2}\left(\frac{1}{AB . AC . AD} + \frac{1}{CD . CA . CB}\right)$$
$$= \frac{1}{4R^2}\left(\frac{CB . CD + AB . AD}{AB . AC . AD . CD . CB}\right)$$
$$= \frac{1}{4R^2} \cdot \frac{\frac{2\,\Delta\, BCD}{\sin C} + \frac{2\,\Delta\, ABD}{\sin A}}{AB . AC . AD . CD . CB}$$

$$= \frac{1}{2R^2 \sin A} \cdot \frac{\text{Area of quadr. ABCD}}{AB \,.\, AC \,.\, AD \,.\, CD \,.\, CB}.$$

Similarly $\dfrac{1}{p_2} + \dfrac{1}{p_4} = \dfrac{1}{2R^2 \sin B} \cdot \dfrac{\text{Area of quadr. ABCD}}{AB \,.\, BC \,.\, CD \,.\, DA \,.\, BD}.$

But $AC = 2R \sin B$ and $BD = 2R \sin A$.

$$\therefore \frac{1}{p_1} + \frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{p_4}.$$

Question 1053.

(A. C. L. WILKINSON) :—Two circles are such that triangles can be inscribed in the one which are circumscribed to the other. If P, Q, R be the points of contact of such a triangle with the circle about which it is circumscribed, show that the nine-point circle of PQR is fixed and that the locus of the centres K of the inscribed or escribed circles of the triangle PQR is the curve $IK^2 = r.\ NK$, where N is the common nine-point centre, I the centre and r the radius of the common circumscribed circle of the triangles PQR.

Solution by Hemraj and N. G. Leather.

It is easy to show that the triangle formed by the tangents at the vertices of a triangle to its circumcircle is similar and similarly situated to the pedal triangle of the latter triangle.

Hence if ABC be any position of a triangle described in a given circle S and about a given circle S′, A′B′C′ the pedal triangle of PQR is similar and similarly situated to $\triangle$ ABC; and hence the loci of the corresponding points of ABC and A′B′C′ are also similar and similarly situated. Hence the circumcentre and incentre of A′B′C′ which are the nine point centre and the orthocentre of PQR are fixed.

Also the radius of the nine point circle = half the radius of S′ = r.

$\therefore$ the nine point circle of PQR is fixed.

Again from the well-known results $IN = \dfrac{R}{2} \pm r$ and $OI^2 = R^2 \pm 2Rr = 2R\,.\,IN$. We have at once $IK^2 = 2r\,.\,NK$,

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Question 1065.

(S. KRISHNASWAMI AIYANGAR) :—If e be the radius of curvature of the curve $r^m = a^m \sin m\theta$ at the point whose distance measured along the curve from a fixed point is s, prove that

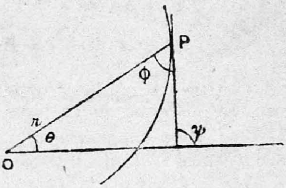
$$(m-1)(m+1)^2\, e\frac{d^2e}{ds^2} - m(m+1)^2 \left(\frac{de}{ds}\right)^2 - m(m-1)^2 = 0.$$

Show how to solve this equation and hence find the equation of the given curve in terms of p and s.

Similar formulæ are got by giving particular values to m, for a parabola, a rectangular hyperbola, a lemniscate and a cardiode.

Solution by N. G. Leather.

$$r^m = a^m \sin m\theta.$$



We know $\phi = m\theta$

and $\therefore\ \psi = (m+1)\theta$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= r\sqrt{1 + \cot^2 m\theta}\ d\theta$$

$$= a \sin^{\frac{1}{m}} m\theta \operatorname{cosec} m\theta\ d\theta$$

$$\therefore\ e = \frac{ds}{d\psi} = \frac{a}{m+1} \sin^{\frac{1}{m}-1} m\theta$$

$$\frac{de}{ds} = \frac{de}{d\theta} \Big/ \frac{ds}{d\theta} = -\frac{(m-1)m}{m(m+1)} \sin^{\frac{1}{m}-2} m\theta \sin^{1-\frac{1}{m}} m\theta \cos m\theta$$

$$= -\frac{(m-1)}{(m+1)} \cot m\theta.$$

$$\therefore \left(\frac{de}{ds}\right)^2 = \left(\frac{m-1}{m+1}\right)^2 \left[\operatorname{cosec}^2 m\theta - 1\right]$$

$$\frac{d^2e}{ds^2} = \frac{\dfrac{d}{d\theta}\left(\dfrac{de}{ds}\right)}{\dfrac{ds}{d\theta}} = \frac{(m-1)m}{m+1} \operatorname{cosec}^2 m\theta\, \frac{1}{a} \sin^{1-\frac{1}{m}} m\theta$$

$$= \frac{m(m-1)}{a(m+1)} \sin^{-1-\frac{1}{m}} m\theta$$

$$\therefore\ e\frac{d^2e}{ds^2} = \frac{m(m-1)}{(m+1)^2} \operatorname{cosec}^2 m\theta.$$

$$\therefore\ (m-1)(m+1)^2 e\frac{d^2e}{ds^2} - m(m+1)^2 \left(\frac{de}{ds}\right)^2 - m(m-1)^2$$

$$= m(m-1)^2 \operatorname{cosec}^2 m\theta - m(m-1)^2[\operatorname{cosec}^2 m\theta - 1] - m(m-1)^2$$

which is zero, and hence the equation.

To solve this write $\dfrac{de}{ds} = p$

and $$\frac{d^2e}{ds^2} = \frac{dp}{de} \cdot \frac{de}{ds} = p\frac{dp}{de}$$

and the equation becomes

$$(m-1)(m+1)^2\, p e\frac{dp}{de} - m(m+1)^2 p^2 - m(m-1) = 0.$$

$$\therefore\ \frac{de}{dp} - e\frac{(m-1)(m+1)^2 p}{m[(m+1)^2 p^2 + (m-1)^2]} = 0$$

which is linear in e.

$$\therefore\ e = Ae^{\frac{m-1}{2m}\log[(m+1)^2 p^2 + (m-1)^2]}$$

$$\therefore\ p^2 = \left(\frac{de}{ds}\right)^2 = Be^{\frac{2m}{m-1}} - \left(\frac{m-1}{m+1}\right)^2.$$

(i) *Parabola* $m = -\frac{1}{2}$, $\left(\frac{de}{ds}\right)^2 = Be^{\frac{2}{3}} - 9$

At vertex $\frac{de}{ds} = 0$, $e = 2a$.

$\therefore$ measuring s from vertex

$$\left(\frac{de}{ds}\right)^2 = 9\left[\left(\frac{e}{2a}\right)^{\frac{2}{3}} - 1\right]$$

$$\int \frac{de}{\left[\left(\frac{e}{2a}\right)^{\frac{2}{3}} - 1\right]^{\frac{1}{2}}} = 3s + \text{const.}$$

$$= \quad 6a\int \sec^3\theta\, d\theta \quad \left(\text{putting } \frac{e}{2a} = \sec^3\theta\right)$$

$$= 6a\left[\sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta\right]$$

$$= 3a\left[\sec\theta\tan\theta + \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right)\right]$$

$$= 3a\left[\left(\frac{e}{2a}\right)^{\frac{1}{3}}\left\{\left(\frac{e}{2a}\right)^{\frac{2}{3}} - 1\right\}^{\frac{1}{2}}\right.$$

$$\left. + \log\frac{\left[\sqrt{\left(\frac{e}{2a}\right)^{\frac{1}{3}} + 1} + \sqrt{\left(\frac{e}{2a}\right)^{\frac{1}{3}} - 1}\right]^2}{2}\right]$$

$$= 3a\left[\left(\frac{e}{2a}\right)^{\frac{1}{3}}\left\{\left(\frac{e}{2a}\right)^{\frac{2}{3}} - 1\right\}^{\frac{1}{2}}\right.$$

$$\left. + \log\left\{\left(\frac{e}{2a}\right)^{\frac{1}{3}} + \sqrt{\left(\frac{e}{2a}\right)^{\frac{2}{3}} - 1}\right\}\right]$$

And when $s = 0$, $e = 2a$ and $\theta = 0$. $\therefore\ s = 0$.

$$\therefore\ s = a\left[\left(\frac{e}{2a}\right)^{\frac{1}{3}}\left\{\left(\frac{e}{2a}\right)^{\frac{2}{3}} - 1\right\}^{\frac{1}{2}} + \log\left\{\left(\frac{e}{2a}\right)^{\frac{1}{3}}\right.\right.$$

$$\left.\left. + \sqrt{\left(\frac{e}{2a}\right)^{\frac{2}{3}} - 1}\right\}\right].$$

(ii) *Rectangular hyperbola* $m = -2$

$$\left(\frac{de}{ds}\right)^2 = Be^{\frac{4}{3}} - 9$$

At vertex, $\frac{de}{ds} = 0,\ e = a.$

$$\therefore \left[\frac{de}{ds}\right]^2 = 9\left[\left(\frac{e}{a}\right)^{\frac{4}{3}} - 1\right].$$

$$\therefore \int \frac{de}{\left[\left(\frac{e}{a}\right)^{\frac{4}{3}} - 1\right]^{\frac{1}{2}}} = 3s + \text{const.}$$

$$\text{or } 3s = \int_a^e \frac{de}{\left[\left(\frac{e}{a}\right)^{\frac{4}{3}} - 1\right]}$$

(iii) *Lemniscate* $m = 2,\ \left(\frac{de}{ds}\right)^2 = Be^4 - \frac{1}{9}$

When $\theta = \frac{\pi}{4},\ \frac{de}{ds} = 0,\ e = \frac{a}{3}.$

$$\therefore B = \frac{9}{a^4}$$

$$\left(\frac{de}{ds}\right)^2 = \frac{1}{9}\left[\left(\frac{3e}{a}\right)^4 - 1\right]$$

$$\therefore \frac{s}{3} = \int_{\frac{a}{3}}^{e} \frac{de}{\left[\left(\frac{3e}{a}\right)^4 - 1\right]^{\frac{1}{2}}}$$

(iv) *Cardiod* $m = \frac{1}{2},\ \left(\frac{de}{ds}\right)^2 = Be^{-2} - \frac{1}{9}$

At $\theta = \pi,\ \frac{de}{ds} = 0,\ e = \frac{2a}{3}.$

$$\therefore \left(\frac{de}{ds}\right)^2 = \left(\frac{2a}{9e}\right)^2 - \frac{1}{9}$$

$$= \frac{1}{9}\left[\left(\frac{2a}{3e}\right)^2 - 1\right]$$

$$\frac{s}{3} = \int_{\frac{2a}{3}}^{e} \frac{de}{\left[\left(\frac{2a}{3e}\right)^2 - 1\right]^{\frac{1}{2}}}$$

$$= -\frac{2a}{3}\int_0^{\cos^{-1}\frac{3e}{2a}} \frac{\sin\theta\, d\theta}{\tan\theta} = -\frac{2a}{3}\int_0^{\cos^{-1}\frac{3e}{2a}} \cos\theta\, d\theta \text{ on putting } 3e = 2a\cos\theta$$

$$= -\frac{2a}{3}\Big[\sin\theta\Big]_0^{\cos^{-1}\frac{3e}{2a}}$$

$$= \frac{2a}{3}\sqrt{1 - \left(\frac{3e}{2a}\right)^2}.$$

$\therefore\ 9e^2 + s^2 = 4a^2.$

Question 1066.

(P. SUBRAMANI AIYAR) :—Solve completely

$$axy\left(\frac{dy}{dx}\right)^2 + (x^2 - ay^2 - b)\frac{dy}{dx} - xy = 0.$$

Solution by N. G. Leather.

Now $\quad axy\left(\dfrac{dy}{dx}\right)^2 + (x^2 - ay^2 - b)\dfrac{dy}{dx} - xy = 0 \qquad \ldots \text{(i)}$

may be written $(ayp + x)(xp - y) - bp = 0.$

Transforming into the related equation by writing

$$\left.\begin{aligned} x &= P \\ p &= X \\ y &= PX - Y \\ px - y &= Y \end{aligned}\right\} \quad \text{(ii)}$$

we get $(aPX^2 - aYX + P)\,Y - bX = 0$

or $(aX^2 + 1)\,PY = X(aY + b)$

which may be integrated.

$$\frac{YdY}{aY + b} = \frac{XdX}{aX^2 + 1}\qquad \left[2 - \frac{2b}{aY + b}\right]dY = \frac{2aXdX}{ax^2 + 1}.$$

$$\therefore\ 2Y - \frac{2b}{a}\log(aY + b) = \log(aX^2 + 1) + c.$$

The complete solution will consist of this, together with

$$axy\,X^2 + (x^2 - ay^2 - b)\,X - xy = 0$$

and

$$xX - y = Y$$

which may be derived from (i) and (ii).

The elimination of X and Y from these three equations will give the complete solution in terms of x and y.

The Singular solution, if it exist, will be

$$(x^2 - ay^2 - b)^2 + 4ax^2y^2 = 0.$$

But this equation satisfies (i), and hence is the Singular solution.

QUESTIONS FOR SOLUTION.

1096. (T. P. TRIVEDI, M.A., L.L.B.) Shew that if p is prime and n any positive integer, $(np!) - n!(p!)^n$ is a multiple of p^{n+3}.

[Extension of a question set in the M.A. examination, 1920, of the Bombay University.]

1097. (T. P. TRIVEDI, M.A., L.L.B.) Prove that

(i) $$\left[\int_0^\infty e^{-x^4} dx\right]^2 = \int_0^\infty e^{-x^2} dx \times \int_0^{\frac{\pi}{2}} \frac{d\theta}{(4-2\sin^2\theta)^{\frac{1}{2}}};$$

(ii) $$\left[\int_0^\infty e^{-x^6} dx\right]^3 = \int_0^\infty e^{-x^3} dx \times \int_0^{\frac{\pi}{2}} \frac{d\theta}{(3-6\sin^2\theta)^{\frac{1}{3}}}.$$

Is it possible to generalise these results?

1098. (K. J. SANJANA, M.A.):—(i) If the circle of curvature at a point of the parabola $y^2 = 4ax$ touches the directrix, the point lies at the intersection of the parabola with the straight line $4x - 5a = 0$.

(ii) If the circle of curvature at a point of the hyperbola $4xy = a^2 + b^2$ touches one of the asymptotes, the point lies at the intersection of the hyperbola with one of the curves

$$3x^4 - 2(a^2 - b^2)x^2 + 6x^2y^2 - y^4 = 0,$$
$$3y^4 - 2(a^2 - b^2)y^2 + 6y^2x^2 - x^4 = 0.$$

1099. (K. J. SANJANA, M.A.):—(i) If n and r are positive integers, prove that ${}^nC_{r-1} - \frac{1}{2}\,{}^nC_{r-2} + \frac{1}{3}\,{}^nC_{r-3} \ldots\ldots + (-1)^{r-1}\frac{1}{r}\,{}^nC_0$

$$= {}^nC_r \left\{\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} \ldots\ldots + \frac{1}{n-r+1}\right\};$$

(ii) If r is a positive integer and n any rational positive number, prove that ${}^nP_{r-1} + \frac{1}{2}\,{}^nP_{r-2} + \frac{1}{3}\,{}^nP_{r-3} \ldots\ldots + \frac{1}{r}\,{}^nP_0$

$$= {}^nP_r \left\{\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} \ldots\ldots + \frac{1}{n+r-1}\right\},$$

where $${}^nP_r = \frac{n(n+1)\ \ldots\ldots\ (n+r-1)}{r!} \text{ and } {}^nP_0 = 1.$$

1100. (LAKSHMISHANKER N. BHATT):—ABC is a triangle having its sides divided internally and externally in the ratios p/q, q/r, r/p. If the four lines joining the points of division, three and three, touch a parabola, then prove that

(i) the parabola can for no values of p/q, q/r, r/p pass through the intersections of the circumcircle and the nine-points circle;

and (ii) if p, q, r are proportional to

$$\sqrt{\{a(\cos A - \cos B\cos C)\}},\ \sqrt{\{b(\cos B - \cos A\cos C)\}},$$
$$\sqrt{\{c(\cos C - \cos A\cos B)\}}$$

then the focus will coincide with the nine-points centre.

In Memoriam.

Mr. S. RAMANUJAN, B.A., F.R.S.

Born, December 1888. Died, April 1920.